The Limit $\lim_{z \to a} \frac{z^n - a^n}{z - a}$ in the Complex Plane

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Abstract

We examine the limit $\lim_{z \to a} \frac{z^n - a^n}{z - a}$ where $z, a, n \in \mathbb{C}$. Unlike the real case, the complex setting introduces fundamental complications: multi-valued functions, branch cuts, path dependence, and holomorphicity conditions. We provide a systematic treatment of all cases and subtleties.

1 Introduction

This article uses a familiar difference quotient to expose why complex exponentiation is more delicate than its real counterpart. The expression is important not because the final formula is surprising, but because reaching it safely requires choosing branches, tracking monodromy, and distinguishing local holomorphic behavior from global multivaluedness.

Contents

2 Preliminaries: Complex Exponentiation

2.1 The Complex Exponential and Logarithm

Definition 2.1 (Complex Exponential)**.** For $z = x + iy \in \mathbb{C}$:

$$e^z = e^x(\cos y + i \sin y). \tag{1}$$

This function is entire (holomorphic on all of $\mathbb{C}$) and periodic with period $2\pi i$.

Definition 2.2 (Complex Logarithm)**.** For $z \in \mathbb{C} \setminus \{0\}$, the **multi-valued logarithm** is:

$$\log z = \ln|z| + i \arg(z) = \ln|z| + i(\operatorname{Arg}(z) + 2\pi k), \quad k \in \mathbb{Z}, \tag{2}$$

where $\operatorname{Arg}(z) \in (-\pi, \pi]$ is the **principal argument**.

Definition 2.3 (Principal Logarithm)**.** The **principal branch** of the logarithm is:

$$\operatorname{Log} z = \ln|z| + i \operatorname{Arg}(z), \quad z \in \mathbb{C} \setminus (-\infty, 0]. \tag{3}$$

This is holomorphic on $\mathbb{C} \setminus (-\infty, 0]$, with branch cut along the negative real axis.

2.2 Complex Powers

Definition 2.4 (Complex Power)**.** For $z, n \in \mathbb{C}$ with $z \neq 0$:

$$z^n = e^{n \log z} = e^{n(\ln|z| + i \arg(z))}. \tag{4}$$

This is generally **multi-valued** due to the multi-valuedness of $\log z$.

Definition 2.5 (Principal Power)**.** Using the principal logarithm:

$$z^n_{\text{principal}} = e^{n \operatorname{Log} z}, \quad z \in \mathbb{C} \setminus (-\infty, 0]. \tag{5}$$

Remark 2.1 (When is z^n single-valued?)*.* The function z^n is single-valued if and only if $n \in \mathbb{Z}$. For $n \notin \mathbb{Z}$, different branches of $\log z$ yield different values of z^n.

3 The Main Limit: Statement and Classification

3.1 Problem Statement

We study:

$$L = \lim_{z \to a} \frac{z^n - a^n}{z - a}, \quad z, a, n \in \mathbb{C}. \tag{6}$$

3.2 Classification of Cases

The analysis depends critically on the parameters:

Case	Condition	Nature
I	$n \in \mathbb{Z}^+$ (positive integer)	Single-valued, entire
II	$n \in \mathbb{Z}^-$ (negative integer)	Single-valued, meromorphic
III	$n = 0$	Trivial
IV	$n \in \mathbb{Q} \setminus \mathbb{Z}$ (rational)	Multi-valued, finite branches
V	$n \in \mathbb{R} \setminus \mathbb{Q}$ (irrational)	Multi-valued, infinite branches
VI	$n \in \mathbb{C} \setminus \mathbb{R}$ (complex)	Multi-valued, infinite branches

For each case, we must also consider the value of a:

- $a = 0$: Special treatment required
- $a \in (-\infty, 0)$: On the branch cut (for principal branch)
- $a \in \mathbb{C} \setminus (-\infty, 0]$: Away from branch cut

4 Case I: Positive Integer Exponent ($n \in \mathbb{Z}^+$)

4.1 General Result

When $n \in \mathbb{Z}^+$, $f(z) = z^n$ is a polynomial, hence entire.

Theorem 4.1. For $n \in \mathbb{Z}^+$ and any $a \in \mathbb{C}$:

$$\lim_{z \to a} \frac{z^n - a^n}{z - a} = na^{n-1}. \tag{7}$$

Proof. The polynomial $z^n - a^n$ has a as a root, so it factors as:

$$z^n - a^n = (z - a)(z^{n-1} + z^{n-2}a + z^{n-3}a^2 + \cdots + a^{n-1}). \tag{8}$$

Thus:

$$\lim_{z \to a} \frac{z^n - a^n}{z - a} = \lim_{z \to a} \sum_{k=0}^{n-1} z^{n-1-k} a^k \tag{9}$$

$$= \sum_{k=0}^{n-1} a^{n-1-k} a^k = \sum_{k=0}^{n-1} a^{n-1} = na^{n-1}. \tag{10}$$

□

4.2 Subcase: $a = 0$

Proposition 4.1. For $n \in \mathbb{Z}^+$ and $a = 0$:

$$\lim_{z \to 0} \frac{z^n - 0}{z - 0} = \lim_{z \to 0} z^{n-1} = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \geq 2. \end{cases} \tag{11}$$

Remark 4.1. This is consistent with $n \cdot 0^{n-1}$: for $n = 1$, we get $1 \cdot 0^0 = 1$ (using the convention $0^0 = 1$); for $n \geq 2$, we get $n \cdot 0 = 0$.

5 Case II: Negative Integer Exponent ($n \in \mathbb{Z}^-$)

Let $n = -m$ where $m \in \mathbb{Z}^+$. Then $z^n = z^{-m} = 1/z^m$.

Theorem 5.1. For $n = -m \in \mathbb{Z}^-$ and $a \in \mathbb{C} \setminus \{0\}$:

$$\lim_{z \to a} \frac{z^{-m} - a^{-m}}{z - a} = -m \cdot a^{-m-1}. \tag{12}$$

Proof.

$$\frac{z^{-m} - a^{-m}}{z - a} = \frac{\frac{1}{z^m} - \frac{1}{a^m}}{z - a} = \frac{a^m - z^m}{z^m a^m (z - a)} \tag{13}$$

$$= -\frac{z^m - a^m}{z - a} \cdot \frac{1}{z^m a^m}. \tag{14}$$

Taking $z \to a$:

$$L = -ma^{m-1} \cdot \frac{1}{a^{2m}} = -ma^{-m-1}. \tag{15}$$

□

5.1 Subcase: $a = 0$

Warning 5.1. For $a = 0$ and $n < 0$, the limit does **not exist** in $\mathbb{C}$:

$$\lim_{z \to 0} \frac{z^{-m} - 0^{-m}}{z - 0} \tag{16}$$

is undefined because $0^{-m} = \frac{1}{0^m}$ is undefined.

6 Case III: Zero Exponent ($n = 0$)

Proposition 6.1. For $n = 0$ and any $a \in \mathbb{C} \setminus \{0\}$:

$$\lim_{z \to a} \frac{z^0 - a^0}{z - a} = \lim_{z \to a} \frac{1 - 1}{z - a} = 0. \tag{17}$$

Remark 6.1. This is consistent with $na^{n-1} = 0 \cdot a^{-1} = 0$.

6.1 Subcase: $a = 0$

Warning 6.1. The expression 0^0 is indeterminate. With the convention $z^0 = 1$ for all z:

$$\lim_{z \to 0} \frac{z^0 - 0^0}{z - 0} = \lim_{z \to 0} \frac{1 - 1}{z} = 0. \tag{18}$$

However, this requires adopting the convention $0^0 = 1$.

7 Case IV: Rational Non-Integer Exponent ($n \in \mathbb{Q} \setminus \mathbb{Z}$)

Let $n = p/q$ where $p \in \mathbb{Z}$, $q \in \mathbb{Z}^+$, $\gcd(|p|, q) = 1$, and $q \geq 2$.

7.1 Multi-Valuedness

Definition 7.1. The q-th roots of z are:

$$z^{1/q} = |z|^{1/q} e^{i(\mathrm{Arg}(z) + 2\pi k)/q}, \quad k = 0, 1, \ldots, q - 1. \tag{19}$$

There are exactly q distinct values.

Proposition 7.1. For $n = p/q$, the function $z^n = z^{p/q}$ has exactly q branches, corresponding to the q choices of $z^{1/q}$.

7.2 Limit on a Fixed Branch

Theorem 7.1. Let $n = p/q \in \mathbb{Q} \setminus \mathbb{Z}$ and let $a \in \mathbb{C} \setminus \{0\}$ not lie on the branch cut. Fix a branch of z^n that is holomorphic in a neighborhood of a. Then:

$$\lim_{z \to a} \frac{z^n - a^n}{z - a} = na^{n-1}, \tag{20}$$

where a^{n-1} is computed using the same branch.

Proof. On any branch where z^n is holomorphic, it is the complex derivative:

$$\frac{d}{dz} z^n = nz^{n-1}. \tag{21}$$

□

7.3 Subcase: a on the Branch Cut

Warning 7.1. If $a \in (-\infty, 0)$ (on the branch cut for the principal branch), the limit requires careful treatment:

- The principal branch z^n is discontinuous at a.
- The limit depends on the **path of approach**.
- One may define a different branch that is continuous at a.

Example 7.1. Let $n = 1/2$, $a = -1$. The principal square root $\sqrt{z}$ has a branch cut on $(-\infty, 0]$.

Approaching $a = -1$ from above ($z = -1 + i\epsilon$, $\epsilon \to 0^+$):

$$\sqrt{-1 + i\epsilon} \to i.$$

Approaching from below ($z = -1 - i\epsilon$, $\epsilon \to 0^+$):

$$\sqrt{-1 - i\epsilon} \to -i.$$

The limits differ, so $\sqrt{z}$ is not continuous at $z = -1$ in the principal branch.

7.4 Subcase: $a = 0$

Proposition 7.2. For $n = p/q$ with $p > 0$ and $a = 0$:

$$\lim_{z \to 0} \frac{z^{p/q}}{z} = \lim_{z \to 0} z^{p/q - 1} = \begin{cases} 0 & \text{if } p/q > 1, \\ 1 & \text{if } p/q = 1 \text{ (but } n \notin \mathbb{Z}\text{, contradiction)}, \\ \text{does not exist} & \text{if } 0 < p/q < 1. \end{cases} \tag{22}$$

Proof. For $0 < p/q < 1$, we have $p/q - 1 < 0$, so $z^{p/q-1} = z^{-(1-p/q)} \to \infty$ as $z \to 0$. □

Warning 7.2. For $p < 0$, the limit at $a = 0$ is undefined since $0^{p/q}$ is undefined for negative p.

8 Case V: Irrational Real Exponent ($n \in \mathbb{R} \setminus \mathbb{Q}$)

Definition 8.1. For $n \in \mathbb{R} \setminus \mathbb{Q}$ and $z \neq 0$:

$$z^n = e^{n \log z} = e^{n(\ln |z| + i \arg(z))}. \tag{23}$$

Since $\arg(z)$ is determined only modulo 2π, and n is irrational, $e^{2\pi i n k}$ takes infinitely many distinct values as k ranges over $\mathbb{Z}$.

Proposition 8.1. For $n \in \mathbb{R} \setminus \mathbb{Q}$, the function z^n has **infinitely many branches**.

Theorem 8.1. Fix a branch of z^n (e.g., the principal branch). For $a \in \mathbb{C}$ with $|a| > 0$ and a not on the branch cut:

$$\lim_{z \to a} \frac{z^n - a^n}{z - a} = n a^{n-1}. \tag{24}$$

Proof. On any branch, $z^n = e^{n \operatorname{Log} z}$ is holomorphic (since $\operatorname{Log} z$ is), and the derivative is $n z^{n-1}$. □

Example 8.1. Let $n = \sqrt{2}$, $a = 1$. Using the principal branch:

$$a^n = 1^{\sqrt{2}} = e^{\sqrt{2} \cdot 0} = 1, \tag{25}$$

$$a^{n-1} = 1^{\sqrt{2}-1} = 1. \tag{26}$$

Thus, $L = \sqrt{2} \cdot 1 = \sqrt{2}$.

9 Case VI: Complex Exponent ($n \in \mathbb{C} \setminus \mathbb{R}$)

9.1 Definition and Multi-Valuedness

Let $n = \alpha + i\beta$ with $\beta \neq 0$.

Definition 9.1.

$$z^n = e^{n \log z} = e^{(\alpha + i\beta)(\ln |z| + i \arg(z))}. \tag{27}$$

Proposition 9.1. For $n = \alpha + i\beta$ with $\beta \neq 0$:

$$z^n = |z|^{\alpha} e^{-\beta \arg(z)} \cdot e^{i(\beta \ln |z| + \alpha \arg(z))}. \tag{28}$$

The factor $e^{-\beta \arg(z)}$ shows that even the **modulus** of z^n depends on $\arg(z)$.

Remark 9.1. Unlike the real case, $|z^n| \neq |z|^{\operatorname{Re}(n)}$ in general; it depends on the argument of z.

9.2 The Limit

Theorem 9.1. For $n \in \mathbb{C}$ and $a \in \mathbb{C} \setminus \{0\}$ not on the branch cut, using a fixed branch:

$$\lim_{z \to a} \frac{z^n - a^n}{z - a} = n a^{n-1}. \tag{29}$$

Example 9.1. Let $n = i$, $a = e$ (Euler's number). Using the principal branch:

$$a^n = e^i = e^{i \operatorname{Log} e} = e^{i \cdot 1} = \cos(1) + i \sin(1), \tag{30}$$

$$a^{n-1} = e^{i-1} = e^{-1} \cdot e^i = \frac{\cos(1) + i \sin(1)}{e}. \tag{31}$$

Thus:

$$L = i \cdot \frac{\cos(1) + i \sin(1)}{e} = \frac{i \cos(1) - \sin(1)}{e}. \tag{32}$$

Example 9.2. Let $n = 1 + i$, $a = i$. Using the principal branch, $\operatorname{Log}(i) = i\pi/2$:

$$a^n = e^{(1+i) \cdot i\pi/2} = e^{i\pi/2 - \pi/2} = e^{-\pi/2} \cdot e^{i\pi/2} = e^{-\pi/2} \cdot i, \tag{33}$$

$$a^{n-1} = e^{i \cdot i\pi/2} = e^{-\pi/2}. \tag{34}$$

Thus:

$$L = (1 + i) \cdot e^{-\pi/2}. \tag{35}$$

10 Corner Cases and Pathologies

10.1 The Point $a = 0$: Summary

Exponent n	**Limit at $a = 0$**	**Explanation**
$n \in \mathbb{Z}^+$, $n = 1$	1	$\lim z^0 = 1$
$n \in \mathbb{Z}^+$, $n \geq 2$	0	$\lim z^{n-1} = 0$
$n \in \mathbb{Z}^-$	Undefined	0^n undefined
$n = 0$	0 (with convention $0^0 = 1$)	
$n \in (0, 1) \cap \mathbb{Q}$	Does not exist	$z^{n-1} \to \infty$
$n \in (1, \infty) \cap \mathbb{Q}$	0	$z^{n-1} \to 0$
$n \in (-\infty, 0) \cap \mathbb{Q}$	Undefined	0^n undefined
$n \in \mathbb{C} \setminus \mathbb{R}$	Undefined/Pathological	0^n undefined

10.2 Path Dependence at Branch Points

Theorem 10.1. Let $f(z) = z^n$ with $n \notin \mathbb{Z}$, and let $a = 0$ or a lie on a branch cut. The limit

$$\lim_{z \to a} \frac{z^n - a^n}{z - a} \tag{36}$$

may depend on the path along which $z \to a$.

Example 10.1 (Path dependence at branch cut). Let $n = 1/2$, $a = -1$ (on the branch cut). Consider paths:

Path 1: $z(t) = -1 + it$, $t \to 0^+$ (from above).

Path 2: $z(t) = -1 - it$, $t \to 0^+$ (from below).

Using the principal branch, the two paths yield different limiting values because $\sqrt{z}$ is discontinuous across the branch cut.

10.3 Essential Singularities

Warning 10.1. At $z = 0$, the function z^n for $n \notin \mathbb{Z}$ has a **branch point**, not an isolated singularity in the usual sense. The behavior near $z = 0$ is more complex than poles or removable singularities.

10.4 The Case $n = a$

Proposition 10.1. If $n = a$ (the exponent equals the base point):

$$\lim_{z \to n} \frac{z^n - n^n}{z - n} = n \cdot n^{n-1} = n^n. \tag{37}$$

10.5 The Case $n = 1/a$ (for $a \neq 0$)

Proposition 10.2. If $n = 1/a$:

$$\lim_{z \to a} \frac{z^{1/a} - a^{1/a}}{z - a} = \frac{1}{a} \cdot a^{1/a-1} = \frac{a^{1/a}}{a^2}. \tag{38}$$

11 Holomorphicity and the Complex Derivative

11.1 Connection to the Derivative

Theorem 11.1. If $f(z) = z^n$ is holomorphic in a neighborhood of $a \neq 0$ (using a fixed branch), then:

$$\lim_{z \to a} \frac{z^n - a^n}{z - a} = f'(a) = na^{n-1}. \tag{39}$$

Proof. This is the definition of the complex derivative. For $z^n = e^{n \operatorname{Log} z}$:

$$\frac{d}{dz} e^{n \operatorname{Log} z} = e^{n \operatorname{Log} z} \cdot \frac{n}{z} = z^n \cdot \frac{n}{z} = nz^{n-1}. \tag{40}$$

□

11.2 When is z^n Holomorphic?

Proposition 11.1. The function $f(z) = z^n$ is holomorphic on $\mathbb{C} \setminus \{0\}$ if and only if $n \in \mathbb{Z}$.

For $n \notin \mathbb{Z}$, z^n is holomorphic only on simply connected domains that avoid 0 and do not encircle 0.

Remark 11.1. The obstruction to global holomorphicity is **monodromy**: continuing z^n around a loop encircling 0 returns a different branch.

12 Conclusion

Theorem 12.1 (Main Result)**.** For $a \in \mathbb{C} \setminus \{0\}$ not on the branch cut, and using a fixed branch of z^n:

$$\boxed{\lim_{z \to a} \frac{z^n - a^n}{z - a} = na^{n-1}, \quad \forall n \in \mathbb{C}.} \tag{41}$$

12.1 Key Caveats

1. **Branch selection:** For $n \notin \mathbb{Z}$, one must fix a branch of z^n.
2. **Branch cuts:** If a lies on a branch cut, the limit is path-dependent or undefined.
3. $a = 0$**:** Requires case-by-case analysis; often undefined or pathological.
4. **Consistency:** a^{n-1} must be computed using the same branch as z^n.
5. **Monodromy:** The result changes if z winds around 0 before approaching a.

12.2 Comparison: Real vs. Complex

Aspect	**Real** ($\mathbb{R}$)	**Complex** ($\mathbb{C}$)
Single-valuedness	For $a > 0$, $n \in \mathbb{R}$	Only for $n \in \mathbb{Z}$
Branch cuts	Not applicable	Required for $n \notin \mathbb{Z}$
Path dependence	Not applicable	At branch points/cuts
Domain of a	$a > 0$ (or $a \neq 0$ for $n \in \mathbb{Z}$)	$a \neq 0$, off branch cuts
Formula	na^{n-1}	na^{n-1} (branch-dependent)