Two Rational Models for a Transient Spherical Signal

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June 8, 2026

Abstract

This note merges two closely related descriptions of a transient spherical signal in three-dimensional space. The first case is a scalar shell model: the signal occupies a time-dependent spherical layer whose outer radius expands asymptotically and whose thickness eventually collapses. The second case is a radial vector field: the signal is represented by a smooth outward field whose magnitude is concentrated near an expanding radius. Both constructions use rational algebraic functions rather than exponentials or Gaussian profiles.

1 Introduction

The models in this article ask how much of a transient expanding signal can be captured with rational functions rather than exponential or Gaussian profiles. This matters because rational models are often easier to inspect algebraically, compare asymptotically, and embed in later symbolic or computational workflows.

2 Common Setting

Let

$$\boldsymbol{r} = (x, y, z) \in \mathbb{R}^3, \qquad r = \|\boldsymbol{r}\| = \sqrt{x^2 + y^2 + z^2}, \qquad t \geq 0.$$

When $r > 0$, the outward radial unit vector is

$$\hat{\boldsymbol{u}}_r = \frac{\boldsymbol{r}}{r}.$$

At $r = 0$, this unit vector is not defined, so any vector-field model below must either be interpreted away from the origin or completed by a convention such as $\boldsymbol{V}(\boldsymbol{0}, t) = \boldsymbol{0}$.

The common physical picture is a signal emitted from the origin, expanding radially, and then dissolving. The two cases differ in how the signal is represented:

- Case I describes the occupied region of space as a scalar shell.
- Case II describes the signal as a smooth radial vector field.

3 Case I: Scalar Finite-Support Shell

In the scalar shell model the signal is present in a spherical layer

$$R_{\text{int}}(t) \le r \le R_{\text{ext}}(t),$$

and absent outside it. The outer radius expands from zero toward a finite limit $R_{\max}$:

$$R_{\text{ext}}(t) = R_{\max}\frac{t}{t+k_1}, \qquad R_{\max} > 0, \quad k_1 > 0. \tag{1}$$

This is a rational saturation law. It begins at 0, increases monotonically, and satisfies

$$\lim_{t\to\infty} R_{\text{ext}}(t) = R_{\max}.$$

3.1 Thickness and Internal Radius

The shell thickness is required to vanish both at the initial instant and at late time. A convenient rational choice is

$$h(t) = R_{\text{ext}}(t)\frac{k_2}{t+k_2} = R_{\max}\frac{t}{t+k_1}\frac{k_2}{t+k_2}, \qquad k_2 > 0. \tag{2}$$

The internal radius is then

$$R_{\text{int}}(t) = R_{\text{ext}}(t) - h(t) = R_{\text{ext}}(t)\left(1 - \frac{k_2}{t+k_2}\right). \tag{3}$$

Thus $R_{\text{int}}(0) = 0$, and for large t the internal radius approaches the outer radius.

3.2 Field Definition

The occupied signal region is

$$\Omega(t) = \left\{\boldsymbol{r} \in \mathbb{R}^3 : R_{\text{int}}(t) \le \|\boldsymbol{r}\| \le R_{\text{ext}}(t)\right\}.$$

A binary scalar signal can be written with an indicator function:

$$S(\boldsymbol{r}, t) = \mathbf{1}_{\Omega(t)}(\boldsymbol{r}). \tag{4}$$

Equivalently, the support is described by

$$(R_{\text{ext}}(t) - h(t))^2 \le x^2 + y^2 + z^2 \le R_{\text{ext}}(t)^2. \tag{5}$$

Substituting (1) and (2), this becomes

$$\left[R_{\max}\frac{t}{t+k_1}\left(1 - \frac{k_2}{t+k_2}\right)\right]^2 \le \|\boldsymbol{r}\|^2 \le \left[R_{\max}\frac{t}{t+k_1}\right]^2. \tag{6}$$

3.3 Behavior

For early time, $t \ll k_1, k_2$,

$$R_{\text{ext}}(t) \approx \frac{R_{\max}}{k_1}t, \qquad h(t) \approx R_{\text{ext}}(t).$$

The internal radius is therefore close to zero, so the signal initially behaves like a solid sphere expanding from the origin.

For late time,

$$R_{\text{ext}}(t) \to R_{\max}, \qquad h(t) \approx R_{\max}\frac{k_2}{t} \to 0.$$

The occupied region becomes thinner and thinner near the limiting sphere $r = R_{\max}$.

4 Case II: Smooth Rational Radial Vector Field

The second model represents the same qualitative phenomenon as a vector field rather than as a sharply supported scalar region. The field is purely radial:

$$\boldsymbol{V}(\boldsymbol{r},t) = M(r,t)\hat{\boldsymbol{u}}_r, \qquad r > 0. \tag{7}$$

Here $M(r,t)$ is a scalar magnitude concentrated near a moving shell radius.

4.1 Rational Shell Magnitude

The shell profile is modeled with a Lorentzian–Cauchy rational bell:

$$M(r,t) = \frac{A(t)}{1 + \left(\dfrac{r-\mu(t)}{w(t)}\right)^2}. \tag{8}$$

The parameters have direct interpretations:

- $\mu(t)$ is the center radius of the shell.
- $w(t)$ is the shell width.
- $A(t)$ is the peak amplitude.

The center radius follows another rational saturation law:

$$\mu(t) = R_{\max}\frac{t}{t+k_{\exp}}, \qquad k_{\exp} > 0. \tag{9}$$

The amplitude decays rationally:

$$A(t) = \frac{I_0\tau}{t+\tau}, \qquad I_0 > 0, \quad \tau > 0. \tag{10}$$

At $t = 0$, the amplitude is I_0, while for large t it decays like $1/t$. A simple diffusive width law is

$$w(t) = w_0 + \beta t, \qquad w_0 > 0, \quad \beta \geq 0. \tag{11}$$

4.2 Full Equation

Combining (7)–(11), the vector field is

$$\boldsymbol{V}(\boldsymbol{r},t) = \left[\frac{\dfrac{I_0\tau}{t+\tau}}{1 + \left(\dfrac{r - R_{\max}\dfrac{t}{t+k_{\exp}}}{w_0+\beta t}\right)^2}\right]\frac{\boldsymbol{r}}{r}, \qquad r > 0. \tag{12}$$

This is not compactly supported: the Lorentzian profile has heavy tails. However, its largest values are concentrated near

$$r = \mu(t),$$

and the peak magnitude is $A(t)$.

4.3 Behavior

The direction is always radial and outward when $M(r,t) > 0$. The denominator in (8) is smallest when $r = \mu(t)$, so the field has a smooth shell-like maximum at the moving radius. The rational decay in $A(t)$ makes the field vanish in amplitude over long times.

The model is algebraically cheap: apart from the norm $r = \|\boldsymbol{r}\|$, it uses only arithmetic operations. In applications where avoiding a square root is important, one may formulate approximate or squared-radius variants, but the radial vector direction itself naturally requires normalization.

5 Comparison of the Two Cases

The scalar shell and vector-field models encode related but distinct ideas.

Feature	Case I	Case II
Quantity	scalar $S(\boldsymbol{r},t)$	vector $\boldsymbol{V}(\boldsymbol{r},t)$
Support	finite shell	infinite heavy-tailed profile
Boundary	sharp inequalities	smooth Lorentzian decay
Radius law	$R_{\text{ext}}(t)$	$\mu(t)$
Dissolution	thickness $h(t) \to 0$	amplitude $A(t) \to 0$
Direction	none	outward radial direction

Case I is appropriate when the main object is the occupied spatial region of the signal. Case II is appropriate when the signal should act as a force, velocity, pressure, or displacement field.

6 Conclusion

Both models describe transient spherical signals using rational functions. The first gives a sharply bounded shell whose thickness grows and then collapses. The second gives a smooth outward radial field whose magnitude forms an expanding rational shell and fades over time. Together they provide two complementary descriptions of the same qualitative phenomenon: a signal born at the origin, expanding through space, and ultimately dissolving.