Input × mirror: an exhaustive classification of points, lines and circles with real, imaginary and complex parameters

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October 2026

Companion to `theory.pdf` and `examples16.pdf`

Abstract

We take every combination of an *input* (a point, a line or a circle) and an *operator* (the mirror), where each defining parameter may be real, purely imaginary or complex, and ask three questions: can the operator act at all, what kind of object comes out, and where do input and operator meet (two real points, a real tangency, a conjugate pair, …). The $45 \times 45 = 2025$ raw parameter combinations collapse in three steps. Under the complex Möbius group $PO(4, \mathbb{C})$ almost everything is one orbit: the only invariant of a pair is κ (and incidence for a point). Under real Möbius maps every object falls into one of seven classes (M1–M5, P1, P2), decided by one real number $\tau = \langle X, \bar{X} \rangle / |Q(X)|$. Pairs fall into 35 cells plus one degenerate rule (points cannot be mirrors), and each cell splits by the type of common points into 88 leaves. Orthogonality is possible in exactly 20 of the 25 circle–circle cells. All exclusions are proved from four short lemmas. All 112 realised leaves are certified by exact examples, and `classification_verify.py` runs 305 checks (0 failures).

Contents

1 Setting and the questions **2**

2 Single objects **2**
- 2.1 Over $\mathbb{C}$: almost no classes . . . 2
- 2.2 Over $\mathbb{R}$: seven classes, one number . . . 3
- 2.3 Collapse of the raw parameter types . . . 4

3 Who can be a mirror, and what comes out **5**

4 Pairs: invariants and four lemmas **6**
- 4.1 Over $\mathbb{C}$ . . . 6
- 4.2 Over $\mathbb{R}$: four lemmas . . . 7
- 4.3 The exact classifier . . . 7

5 Real–real pairs: what Euclid sees and Möbius forgets **8**

6 The full grid **8**

7 Counting the collapse **12**

8 Surprises **12**

9 Verification map and honesty notes **13**

1 Setting and the questions

We use the notation of `theory.pdf`. A generalized circle is $X = [a : b : c : d] \in \mathbb{P}^3(\mathbb{C})$, read as $a(x^2 + y^2) + bx + cy + d = 0$; it is a *line* if $a = 0$. We write

$$\langle X, Y\rangle = bb' + cc' - 2(ad' + a'd), \qquad Q(X) = \langle X, X\rangle = b^2 + c^2 - 4ad,$$

$\mathrm{pt}(p) = [1 : -2x_0 : -2y_0 : x_0^2 + y_0^2]$ for the point-circle at $p = (x_0, y_0) \in \mathbb{C}^2$, and $\infty = [0 : 0 : 0 : 1]$. Incidence is orthogonality: $\langle \mathrm{pt}(p), X\rangle = -2X(p)$, and $\langle X, \infty\rangle = -2a$, so every line passes through ∞. A *point* is a vector with $Q = 0$; for $a \neq 0$ it is $\mathrm{pt}(p)$, and for $a = 0$ it is ∞ or an *isotropic line* $[0 : 1 : \pm i : \delta]$ (the non-real "points at infinity" of the quadric $\mathcal{Q} = \{Q = 0\}$).

The mirror. For $Q(L) \neq 0$, $C \star L = C - 2\langle C, L\rangle L/Q(L)$. The pair invariant is

$$\kappa(C, L) = \frac{\langle C, L\rangle^2}{Q(C)Q(L)} \qquad (Q(C)Q(L) \neq 0).$$

It is $\cos^2$ of the angle for real secant circles, 1 for tangency and 0 for orthogonality. The common points of C and L are the null vectors orthogonal to both. Counted on $\mathcal{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1$ there are always two, or one double, unless $C = L$; and $C \cap (C \star L) = C \cap L$ whenever $\langle C, L\rangle \neq 0$ (`theory.pdf`, Thm. 5.1).

Raw parameter types. A point (x_0, y_0) has two parameters. A line $y = mx + q$, i.e. $[0 : m : -1 : q]$, has (m, q); vertical lines are equivalent to $m = 0$ by a rotation and give nothing new. A circle $(x - m_1)^2 + (y - m_2)^2 = r^2$ has (m_1, m_2, r). Each parameter is of type `R` (real), `I` (purely imaginary, $\neq 0$) or `C` (both parts $\neq 0$). This gives $9 + 9 + 27 = 45$ types per role, and $45^2 = 2025$ ordered (input, operator) types.

Outcome codes for the two common points of a pair:

O1 two real points	**O2** one real double point	**O3** a conjugate pair $p, \bar{p}$
O4 one real, one non-real	**O5** one non-real double point	**O6** two non-real, not conjugate

"Real" means real on $\mathcal{Q}$: a real finite point, or ∞. Over $\mathbb{C}$ only "two distinct" (O1, O3, O4, O6) and "double" (O2, O5) can be told apart; realness is what splits them. A superscript $\perp$ marks an orthogonal pair ($\kappa = 0$).

Verification. Each claim carries a tag such as [L2], the tag of the matching check in `verify/classification_verify.py`. That script writes `verify/classification_verify_log.txt` and `classification/classification_data.json`; all tables below are generated from the JSON by `classification/classification_build.py`. Exact checks use Gaussian rationals $\mathbb{Q}(i)$; numerical root checks use 80-digit mpmath. Random checks are evidence. The impossibility statements are *proved* in the text; the script only confirms that random and targeted search never contradicts them.

2 Single objects

2.1 Over $\mathbb{C}$: almost no classes

Proposition 2.1. *(a) $PO(4, \mathbb{C})$ has exactly two orbits on $\mathbb{P}^3(\mathbb{C})$: $\{Q \neq 0\}$ (circles and lines alike) and $\{Q = 0\}$ (points, ∞ and isotropic lines alike). (b) The complex similarities (the stabiliser of ∞) have exactly five orbits: proper circles ($a \neq 0, Q \neq 0$), non-isotropic lines ($a = 0, Q \neq 0$), finite points, isotropic lines, and $\{\infty\}$.*

Proof. (a) By Witt's theorem [1], for a non-degenerate quadratic form over $\mathbb{C}$ any two vectors with the same non-zero value of Q are related by an isometry. Scaling makes $Q = 1$, which gives one orbit. All non-zero null vectors also form one orbit. (b) Complex translations, rotations $SO(2, \mathbb{C})$, reflections and complex scalings send any circle $(x - m)^2 = \rho$ with $\rho \neq 0$ to $x^2 + y^2 = 1$. They send any line with $b^2 + c^2 \neq 0$ to $x = 0$, and any point to $(0, 0)$. They send $[0 : 1 : \pm i : \delta]$ to $[0 : 1 : i : 0]$: use a reflection for the sign and a translation to remove δ. These classes are distinct because $a = 0$ and $Q = 0$ are both preserved. □

So over $\mathbb{C}$ the parameter types (real, imaginary, complex) carry no information at all: $x^2+y^2=1$, $x^2+y^2=-1$ *and* $(x-i)^2+y^2=2i$ *are the same object.* The only invariant is whether Q vanishes.

2.2 Over $\mathbb{R}$: seven classes, one number

Real Möbius geometry is the group $PO(3,1)$ of real matrices preserving Q. It commutes with conjugation $X \mapsto \bar{X}$. Write $X = R + iS$ with R, S real, and let $W_R(X) = \operatorname{span}_{\mathbb{R}}(R, S)$. Define

$$h(X) = \langle X, \bar{X} \rangle = Q(R) + Q(S) \in \mathbb{R}, \qquad \tau(X) = \frac{h(X)}{|Q(X)|} \quad (Q(X) \neq 0).$$

Both are projective invariants: $X \mapsto \lambda X$ multiplies h and $|Q|$ by $|\lambda|^2$. They are also invariant under real Möbius maps.

Theorem 2.2 (Seven real classes)**.** *Every object belongs to exactly one class in the table, and the number of its real points is as listed.*

class	*definition*	τ	*real points*	*normal form / examples*
M1	*real,* $Q > 0$	1	*a real circle/line*	$x^2+y^2=1$, $y=0$
M2	*real,* $Q < 0$	-1	*none*	$x^2+y^2=-1$
M3	*non-real,* W_R *positive definite*	> 1	*two*	$x+2iy=0$; $(x-2i)^2+y^2=-1$
M4	*non-real,* W_R *degenerate*	$= 1$	*one*	$R + t\,\mathrm{pt}(r)$, R *real,* $r \in R$ *real,* $t \notin \mathbb{R}$; $x = i$
M5	*non-real,* W_R *Lorentzian*	$\in (-1,1)$	*none*	$x^2+y^2=i$; $(x-\frac{i}{2})^2+y^2=-1$
P1	*real point* $(Q=0)$	–	*itself*	$\mathrm{pt}(0,0)$, ∞
P2	*non-real point* $(Q=0)$	–	*its Laguerre pair*	$\mathrm{pt}(i,0)$, *isotropic lines*

Proof. A real point p lies on X iff $\langle \mathrm{pt}(p), R\rangle = \langle \mathrm{pt}(p), S\rangle = 0$, i.e. iff the real null vector $\mathrm{pt}(p)$ lies in the real plane $W_R^{\perp}$. When X is non-real, W_R is two-dimensional and, since Q has signature $(3,1)$, $W_R^{\perp}$ has the complementary type. A positive definite W_R gives a Lorentzian $W_R^{\perp}$, which contains two real null lines. A degenerate W_R gives one, and a Lorentzian W_R gives a definite $W_R^{\perp}$, which contains none.

For τ: $Q(X) = Q(R) - Q(S) + 2i\langle R, S\rangle$, so

$$h^2 - |Q|^2 = 4\big(Q(R)Q(S) - \langle R, S\rangle^2\big) = 4 \det \mathrm{Gram}(R, S).$$

Hence $|\tau| > 1, = 1, < 1$ according to the three cases. A negative definite real plane does not exist in signature $(3,1)$. A degenerate plane lies in $n^{\perp}$ for its null radical n, and $n^{\perp}$ is positive semi-definite. So $h > 0$ in the first two cases, which gives $\tau > 1$ and $\tau = 1$. The M4 normal form: the radical is a real null vector, i.e. a real point r, and $W_R = \operatorname{span}(R, \mathrm{pt}(r))$ with $R \perp \mathrm{pt}(r)$. □

Two facts make τ easy to read. First, $\kappa(X, \bar{X}) = \tau^2$, so τ is (up to sign) the angle invariant between an object and its conjugate. Second, X and $\bar{X}$ meet in O1, O2, O3 for M3, M4, M5 respectively, and in O1 for P2. [S2]

Proposition 2.3 (Non-real points)**.** *[S2] For* $p = u + iw \in \mathbb{C}^2$ *with* $u, w \in \mathbb{R}^2$, $w \neq 0$*, the two isotropic lines through* p *meet the real plane in the* Laguerre pair $u \mp Jw$*, where* J *is the rotation by* $90°$*. The pair determines* p *up to conjugation. Real similarities act transitively on non-real finite points, so every one of them is* $(i, 0)$ *in suitable real coordinates. The isotropic lines, i.e. the non-real points of* $\mathcal{Q}$ *at infinity, belong to the same real Möbius class* P2*. Inverting* p *in* any *real circle through its Laguerre pair gives* $\bar{p}$*.*

Proof of the last claim. On $\mathcal{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1$ write $p = (\alpha, \beta)$; the real points are $(z, \bar{z})$ and $\bar{p} = (\bar{\beta}, \bar{\alpha})$. The real points on the rulings through p are $q_1 = (\alpha, \bar{\alpha})$ and $q_2 = (\bar{\beta}, \beta)$, the Laguerre pair. Inversion in a real circle acts as $(u, v) \mapsto (f(v), \bar{f}(u))$ for a Möbius f. Fixing q_1 and q_2 gives $\bar{f}(\alpha) = \bar{\alpha}$ and $f(\beta) = \bar{\beta}$, so $p \mapsto (\bar{\beta}, \bar{\alpha}) = \bar{p}$. □

2.3 Collapse of the raw parameter types

Proposition 2.4 (Explicit criteria)**.** *[S1] (a) A point is P1 iff both coordinates are real. (b) For the line $y = mx + q$: $h = 1 + |m|^2$, $Q = 1 + m^2$, and $h^2 - |Q|^2 = 4(\operatorname{Im} m)^2$. It is M1 if $m, q \in \mathbb{R}$, M4 if $m \in \mathbb{R}$, $q \notin \mathbb{R}$ (its one real point is ∞), isotropic (P2) if $m = \pm i$, and M3 for every other non-real m (its real points are ∞ and one finite point).* A line is never M5*, since it passes through the real point ∞. (c) For the circle with centre $m = u + iw$ and $\rho = r^2$: $h = 8|w|^2 + 4 \operatorname{Re} \rho$, $Q = 4\rho$, and*

$$h^2 - |Q|^2 = 16\,\Delta, \qquad \Delta = 4|w|^2(\operatorname{Re}\rho + |w|^2) - (\operatorname{Im}\rho)^2.$$

It is M3, M4, M5 as $\Delta > 0, = 0, < 0$ (if non-real). Equivalently, its real points are those of the real circle $|p - u|^2 = \operatorname{Re}\rho + |w|^2$ on the real line $2(p - u) \cdot w = -\operatorname{Im}\rho$. In particular, a real centre with $r \in \mathbb{R}$, $i\mathbb{R}$, or complex gives M1, M2, M5 ($\tau = \cos \arg \rho$). A non-real centre with a real radius always gives M3.

Proof. (b) and (c) follow by expanding $X(p) = 0$ for real p into real and imaginary parts, and by the identity in Theorem 2.2. For lines, $|1 + m^2| \le 1 + |m|^2$ with equality iff $m^2 \ge 0$. □

Table 1: The 45 raw types and the classes they reach (I=imaginary, C=complex; for circles the code is $m_1 m_2 r$). "iso" = isotropic line. Every listed class is realised: by random sampling, plus the explicit M4 witnesses of [S1] where sampling missed them (e.g. $(x - i)^2 + (y - i)^2 = 4i$ for `III`).

object	code	classes
point	`RR`	P1
point	`RI`	P2
point	`RC`	P2
point	`IR`	P2
point	`II`	P2
point	`IC`	P2
point	`CR`	P2
point	`CI`	P2
point	`CC`	P2
line	`RR`	M1
line	`RI`	M4
line	`RC`	M4
line	`IR`	M3, iso
line	`II`	M3, iso
line	`IC`	M3, iso
line	`CR`	M3
line	`CI`	M3
line	`CC`	M3
circle	`RRR`	M1
circle	`RRI`	M2
circle	`RRC`	M5
circle	`RIR`	M3
circle	`RII`	M3, M4, M5
circle	`RIC`	M3, M4, M5
circle	`RCR`	M3
circle	`RCI`	M3, M4, M5
circle	`RCC`	M3, M4, M5
circle	`IRR`	M3
circle	`IRI`	M3, M4, M5
circle	`IRC`	M3, M4, M5
circle	`IIR`	M3
circle	`III`	M3, M4, M5

object	code	classes
circle	IIC	M3, M4, M5
circle	ICR	M3
circle	ICI	M3, M4, M5
circle	ICC	M3, M4, M5
circle	CRR	M3
circle	CRI	M3, M4, M5
circle	CRC	M3, M4, M5
circle	CIR	M3
circle	CII	M3, M4, M5
circle	CIC	M3, M4, M5
circle	CCR	M3
circle	CCI	M3, M4, M5
circle	CCC	M3, M4, M5

Collapse. The 9 point types become 2 classes: RR gives P1, the other 8 give P2. The 9 line types reach 3 classes plus isotropic lines (P2). The 27 circle types reach all of M1–M5. Only RRR, RRI, RRC (real centre) and the 8 types xyR with a non-real centre and a real radius have a single class. The remaining 16 types depend on the actual values through the sign of Δ.

3 Who can be a mirror, and what comes out

Proposition 3.1 (Points are not mirrors)**.** *[D1] If $Q(L) = 0$ (P1, P2, ∞, an isotropic line) the formula is undefined. The cleared map $X \mapsto Q(L)X - 2\langle X, L\rangle L = -2\langle X, L\rangle L$ is the constant map to L (and zero on circles through L), so it is not a projective transformation. Limits do not help: for $L_\varepsilon : x^2 + y^2 = \varepsilon^2$,*

$$C \star L_\varepsilon \to \begin{cases} \mathrm{pt}(0) & C \not\ni 0, \\ \text{the tangent line of } C \text{ at } 0 & C \ni 0, \end{cases}$$

whereas other families shrinking to the same point, e.g. $x^2 + y^2 + \varepsilon(x - 1) = 0$, give a different limit. So "reflecting in a point" has no well-defined meaning, and the point rows of the operator column collapse to one degenerate rule.

Proof. With $L_\varepsilon = [1 : 0 : 0 : -\varepsilon^2]$ and $C = [a : b : c : d]$: $Q(L_\varepsilon) = 4\varepsilon^2$ and $\langle C, L_\varepsilon\rangle = 2a\varepsilon^2 - 2d$, so $C \star L_\varepsilon = C - (a - d/\varepsilon^2)[1 : 0 : 0 : -\varepsilon^2]$. If $d \neq 0$, multiplying by ε^2 gives the limit $[d : 0 : 0 : 0] = \mathrm{pt}(0)$. If $d = 0$, the result is $[0 : b : c : a\varepsilon^2] \to [0 : b : c : 0]$, which is $bx + cy = 0$. □

Proposition 3.2 (Real mirrors preserve class)**.** *[M1] If L is M1 or M2, then s_L is a real Möbius map and $C \star L$ has the class of C.*

Proposition 3.3 (Non-real mirrors and real circles)**.** *[M2] Let $C \neq C'$ be real circles. A mirror L with $C \star L = C'$ lies in* span(C, C') *and equals $C \pm \mu C'$ with $\mu^2 = Q(C)/Q(C')$. It is non-real iff $Q(C)Q(C') < 0$, i.e. one is M1 and the other M2. In that case $h(L) = 0$, i.e. $\tau(L) = 0$, and L is M5. Example: $x^2 + y^2 = \pm i$ swaps $x^2 + y^2 = 1$ and $x^2 + y^2 = -1$.*

Proof. $C \star L \propto C'$ forces $L \propto C - \lambda C'$. Comparing Q of both sides of $s_L(C) = \lambda C'$ gives $\lambda^2 = Q(C)/Q(C')$. If the ratio is negative, $\mu = it$ with $t \in \mathbb{R}$, and $h(C \pm itC') = Q(C) + t^2 Q(C') = 0$. □

Proposition 3.4 (Line or circle?)**.** *[M3] $C \star L$ passes through ∞ (it is a line, an isotropic line or ∞) iff C passes through $\infty \star L$. For a proper circle L, $\infty \star L$ is its centre (also for a complex centre); for a line L, $\infty \star L = \infty$. So reflection in a circle turns C into a line iff C passes through the centre of L. Reflection in a line turns lines into lines and circles into circles.*

Table 2: Classes of $C \star L$ observed for input class × mirror class (exact arithmetic, 305-check run; [I1]). The M1/M2 columns are proved (the class is preserved). The other columns are *observed lower bounds*, not a proven classification.

input	mirror	image classes
M1	M1	M1
M1	M2	M1
M1	M3	M1, M3
M1	M4	M1, M4
M1	M5	M1, M2, M4, M5
M2	M1	M2
M2	M2	M2
M2	M3	M2, M4
M2	M5	M1, M4, M5
M3	M1	M3
M3	M2	M3
M3	M3	M3, M5
M3	M4	M3
M3	M5	M1, M2, M3, M4, M5
M4	M1	M4
M4	M2	M4
M4	M3	M2, M4
M4	M4	M1, M2, M4
M4	M5	M1, M2, M4
M5	M1	M5
M5	M2	M5
M5	M3	M4, M5
M5	M4	M5
M5	M5	M1, M4, M5
P1	M1	P1
P1	M2	P1
P1	M3	P1, P2
P1	M4	P1, P2
P1	M5	P1, P2
P2	M1	P2
P2	M2	P2
P2	M3	P1, P2
P2	M4	P1, P2
P2	M5	P1, P2

4 Pairs: invariants and four lemmas

4.1 Over $\mathbb{C}$

Proposition 4.1 (Complex orbit types)**.** *Ordered pairs (C, L) with $Q(L) \neq 0$ fall into the following $PO(4, \mathbb{C})$-orbit types: (i) $C = L$; (ii) $Q(C) \neq 0$, $C \neq L$, $\kappa = 1$ (tangent: one double common point); (iii) $Q(C) \neq 0$, $\kappa \neq 1$: one orbit for each value of $\kappa \in \mathbb{C} \setminus \{1\}$ (two distinct common points), with $\kappa = 0$ (orthogonal, $C \star L = C$) singled out; (iv) C a point on L; (v) C a point off L. That gives six types if $\kappa = 0$ is counted separately, and κ is the only continuous invariant.*

Proof. For (iii), the Gram matrix of (C, L) has determinant $Q(C)Q(L)(1 - \kappa) \neq 0$. Over $\mathbb{C}$ one can scale to $Q(C) = Q(L) = 1$, and $\langle C, L \rangle = \pm\sqrt{\kappa}$; the sign is absorbed by $C \mapsto -C$. Witt's extension theorem [1] then gives an isometry between any two such pairs. Witt's theorem also covers the degenerate spans in (ii), (iv) and (v). This is a quoted theorem, not checked by the script. □

So, over $\mathbb{C}$, all of "secant", "disjoint", "nested", "separate", "concentric" and "line missing the circle" are the *same* relation "$\kappa \neq 1$". "Tangent" and "parallel lines" are both $\kappa = 1$, and "through a point" is (iv).

4.2 Over $\mathbb{R}$: four lemmas

Lemma 4.2 (Real pencil)**.** *[L1] Suppose* $\operatorname{span}(C, L)$ *is real (equal to its conjugate). Then the outcome is O1, O2 or O3, decided by the real pencil's signature (Lorentzian, degenerate, definite). If the pencil contains a non-real member X, the outcome is O1, O2, O3 for X of class M3/P2, M4, M5.*

Proof. The common points form a conjugation-invariant set. A double point p then has $\bar{p}$ double too, so p is real. If $X = R + iS$ is in the pencil, the pencil is $W_R(X)$; apply Theorem 2.2. □

Lemma 4.3 (Non-real pencil)**.** *[L2] If* $\operatorname{span}(C, L)$ *is not real, there is at most one real common point, and never a conjugate pair. So the outcome is O2 or O4 (with a real point) or O5 or O6 (without).*

Proof. A real common point is orthogonal to the pencil and its conjugate, whose sum has dimension at least 3. So there is at most one. If p and $\bar{p}$ were both common points, the pencil would be $\operatorname{span}(p, \bar{p})^{\perp}$, which is real. □

Lemma 4.4 (Tangency)**.** *[S3, L3] (a) If C and L are tangent at a real point, then $\tau(C) = \tau(L)$. (b) If C is M1 and L is tangent to it at a non-real point, then L is M3. (c) If C is M2, a circle tangent to C at a non-real point can be M3, M4 or M5. Example: $x^2 + y^2 = -1$ and $x^2 + y^2 - ix = 0$ (M4) touch at $(i, 0)$.*

Proof. The circles tangent to C at p are $L = C + t\,\mathrm{pt}(p)$ with $\langle C, \mathrm{pt}(p)\rangle = 0$, so $Q(L) = Q(C)$. (a) If p is real, the cross terms of $h(L)$ vanish and $h(L) = h(C)$. (b) If C is real, the cross terms vanish again and $h(L) = Q(C) + |t|^2 h(\mathrm{pt}(p))$. For a non-real null p we have $h(\mathrm{pt}(p)) > 0$: write $\mathrm{pt}(p) = R' + iS'$, so that $Q(R') = Q(S')$ and $R' \perp S'$, which forces $Q(R') > 0$. Hence $\tau(L) > 1$, and L is non-real. (c) $\tau(L) = -1 + |t|^2 h(\mathrm{pt}(p))/|Q(C)|$ takes every value in $(-1, \infty)$. □

Lemma 4.5 (κ ranges and orthogonality)**.** *[K1] For real pairs: $\kappa(M1, M1) \geq 0$ and $\kappa(M1, M2) \leq 0$. Also $\kappa(M2, M2) > 1$ for $C \neq L$, so* two imaginary-radius circles are never tangent and never orthogonal. *Orthogonality is impossible in exactly the cells M2×M2, M2×M4, M4×M2, M2×M5, M5×M2.*

Proof. A real plane containing a timelike vector ($Q < 0$) is Lorentzian, so its Gram determinant $Q(C)Q(L)(1 - \kappa)$ is negative. An M4 mirror $R + t\,\mathrm{pt}(r)$ is orthogonal to a real C iff $C \perp R$ and $C \perp \mathrm{pt}(r)$, and no timelike vector is orthogonal to a null one. An M5 mirror is orthogonal to a real C iff $C \perp W_R$, but $W_R^{\perp}$ is then positive definite. In the other 20 cells an orthogonal example exists (Table 6). □

The script also confirms these ranges numerically. The observed sampled ranges are $\kappa \in [0, 1024]$ for M1M1, $[-62.5, 0]$ for M1M2, and $[1.09, 32]$ for M2M2.

4.3 The exact classifier

`classification_core.outcome` decides the outcome in exact arithmetic. If $\operatorname{rank}\{C, L, \bar{C}, \bar{L}\} = 2$, the pencil is real and the signature of a real basis gives O1/O2/O3. If the rank is 3, the pencil plus its conjugate has a real normal n: if $Q(n) = 0$, then n is the real common point and the outcome is O2 or O4; otherwise there is none and the Gram determinant separates O5 (double) from O6. It agrees with 80-digit numerical roots on 1196 random pairs. [L0]

5 Real–real pairs: what Euclid sees and Möbius forgets

Normalise proper circles to $a = 1$. Then $\langle C_1, C_2\rangle = 2(r_1^2 + r_2^2 - |m_1 - m_2|^2)$. Its sign separates disjoint circles: *separate* iff $\langle C_1, C_2\rangle < 0$, *nested* iff > 0. Similarly, a real point is inside C iff $\langle \mathrm{pt}(p), C\rangle > 0$. [K2] Both signs flip under $C \mapsto -C$, so they are *not* projective invariants. Möbius maps exchange inside and outside: an inversion turns a nested pair into a separate pair with the same κ. Concentric circles are just disjoint circles placed symmetrically. Hence:

Table 3: The Euclidean relations Carlo listed, with κ, the sign of $\langle C, L\rangle$ (at $a = 1$), the outcome, the number of finite common points, and the kind of $C \star L$. [E1]

pair	relation	C	L	κ	$\langle C, L\rangle$	O	fin.	$C \star L$
circle–circle	secant	$x^2{+}y^2 - 1 = 0$	$x^2{+}y^2 - 2x = 0$	$\frac{1}{4}$	2	1	2	line
circle–circle	orthogonal	$x^2{+}y^2 - 1 = 0$	$x^2{+}y^2 - 4x + 1 = 0$	0	0	1	2	circle
circle–circle	externally tangent	$x^2{+}y^2 - 1 = 0$	$x^2{+}y^2 - 4x + 3 = 0$	1	-4	2	2	circle
circle–circle	internally tangent	$x^2{+}y^2 - 1 = 0$	$x^2{+}y^2 - x = 0$	1	2	2	2	circle
circle–circle	separate	$x^2{+}y^2 - 1 = 0$	$x^2{+}y^2 - 6x + 8 = 0$	$\frac{49}{4}$	-14	3	2	circle
circle–circle	nested	$x^2{+}y^2 - 1 = 0$	$x^2{+}y^2 - x + \frac{3}{16} = 0$	$\frac{169}{64}$	13/8	3	2	circle
circle–circle	concentric	$x^2{+}y^2 - 1 = 0$	$x^2{+}y^2 - 4 = 0$	$\frac{25}{16}$	10	3	0	circle
circle–line	secant	$x^2{+}y^2 - 1 = 0$	$2x - 1 = 0$	$\frac{1}{4}$	2	1	2	circle
circle–line	through the centre	$x^2{+}y^2 - 1 = 0$	$x = 0$	0	0	1	2	circle
circle–line	tangent	$x^2{+}y^2 - 1 = 0$	$x - 1 = 0$	1	2	2	2	circle
circle–line	missing (Quilez)	$x^2{+}y^2 - 1 = 0$	$x - 2 = 0$	4	4	3	2	circle
line–circle	secant	$2x - 1 = 0$	$x^2{+}y^2 - 1 = 0$	$\frac{1}{4}$	2	1	2	circle
line–circle	through the centre	$x = 0$	$x^2{+}y^2 - 1 = 0$	0	0	1	2	line
line–circle	tangent	$x - 1 = 0$	$x^2{+}y^2 - 1 = 0$	1	2	2	2	circle
line–circle	missing	$x - 2 = 0$	$x^2{+}y^2 - 1 = 0$	4	4	3	2	circle
line–line	crossing at 45 deg	$y = 0$	$x - y = 0$	$\frac{1}{2}$	-1	1	1	line
line–line	perpendicular	$y = 0$	$x = 0$	0	0	1	1	line
line–line	parallel	$y = 0$	$y - 1 = 0$	1	1	2	0	line

Table 4: A real point (P1) against a real circle or line. [E2]

relation	point	L	$\langle \mathrm{pt}, L\rangle$	O	$\mathrm{pt} \star L$
inside	$\mathrm{pt}(\frac{1}{2}, 0)$	$x^2{+}y^2 - 1 = 0$	3/2	3	$\mathrm{pt}(2, 0)$
outside	$\mathrm{pt}(2, 0)$	$x^2{+}y^2 - 1 = 0$	-6	3	$\mathrm{pt}(\frac{1}{2}, 0)$
centre	$\mathrm{pt}(0, 0)$	$x^2{+}y^2 - 1 = 0$	2	3	∞
on the circle	$\mathrm{pt}(1, 0)$	$x^2{+}y^2 - 1 = 0$	0	2	$\mathrm{pt}(1, 0)$
on a line	$\mathrm{pt}(0, 0)$	$x = 0$	0	2	$\mathrm{pt}(0, 0)$
off a line	$\mathrm{pt}(1, 0)$	$x = 0$	-2	3	$\mathrm{pt}(-1, 0)$
infinity vs circle	∞	$x^2{+}y^2 - 1 = 0$	-2	3	$\mathrm{pt}(0, 0)$
infinity vs line	∞	$x = 0$	0	2	∞

Collapse. The 18 real relations (circle/line, in either order) collapse to four κ-classes: $0 < \kappa < 1$ (secant), $\kappa = 0$ (orthogonal, which includes "line through the centre" and "perpendicular"), $\kappa = 1$ (tangent internally or externally, and *parallel lines*, tangent at ∞), and $\kappa > 1$ (separate, nested, concentric, a line missing a circle). These are three outcome codes O1, O2, O3. The 8 point relations collapse to two: on (O2) and off (O3; inside, outside, centre and "∞ vs. a circle" are all the same). For real points against a real circle, "off" always means the two isotropic lines through p cut the circle in a conjugate pair. Tangency of real circles has the extra Euclidean split internal/external (sign of $\langle C, L\rangle$), which is also lost projectively.

6 The full grid

Theorem 6.1 (The grid)**.** *For input class (row) and mirror class (column), the possible outcomes are exactly those in Table 5. A $\perp$ marks outcomes also realised by an orthogonal pair.*

input \ mirror	M1	M2	M3	M4	M5
M1	$1^{\perp}$,2,3	$3^{\perp}$	1,4,5,$6^{\perp}$	2,$4^{\perp}$,6	3,$6^{\perp}$
M2	$3^{\perp}$	3	5,$6^{\perp}$	5,6	3,5,6
M3	1,4,5,$6^{\perp}$	5,$6^{\perp}$	$1^{\perp}$,2,$4^{\perp}$,5,$6^{\perp}$	4,5,$6^{\perp}$	5,$6^{\perp}$
M4	2,$4^{\perp}$,6	5,6	4,5,$6^{\perp}$	2,$4^{\perp}$,5,$6^{\perp}$	5,$6^{\perp}$
M5	3,$6^{\perp}$	3,5,6	5,$6^{\perp}$	5,$6^{\perp}$	$3^{\perp}$,5,$6^{\perp}$
P1	2,3	3	2,6	2,6	3,6
P2	1,4,5,6	5,6	1,4,5,6	4,5,6	5,6
point as mirror	not a mirror ($Q(L) = 0$), Prop. 3.1				

Table 5: Outcome codes (O1–O6 written as 1–6) per cell. There are 88 leaves (cell, outcome), and 24 of them are also realised orthogonally, spread over 20 cells. [G0, G2]

Proof. Exclusions. Lemma 4.2 and Lemma 4.3 restrict to O1–O3 for real pencils and O2, O4–O6 otherwise. A real pencil with a non-real member has its outcome fixed by that member's class, so a real pencil between two non-real objects of different classes is impossible (for example in M3×M4 and M3×M5). Objects without real points (M2, M5) exclude O1, O2 and O4. A real double point needs equal τ (Lemma 4.4a), which kills O2 in, e.g., M1×M3, M1×M5 and M3×M4. Tangency to M1 at a non-real point forces M3 (Lemma 4.4b), which kills O5 in M1×M4, M1×M5 and their transposes. For a point input p: the common points of $\mathrm{pt}(p)$ and L are one point on each isotropic line through p. For real p these are non-real unless $p \in L$, so P1 gives O2 (on), O3 (real pencil) or O6. For non-real p, a real common point is a Laguerre point of p. *Realisations.* Each of the 88 leaves and 24 orthogonal sub-leaves has an exact example in Table 6. Each example is checked for its exact outcome, the 80-digit numerical roots, both classes, involutivity of $\star$, $Q(C \star L) = Q(C)$, $\kappa(C \star L, L) = \kappa(C, L)$, incidence of the listed points on both curves, and $C \cap (C \star L) = C \cap L$. [G1] □

Reading the grid. The M1 row is Carlo's "real circle or line against anything". Against a real mirror it gives the three familiar cases. Against a mirror with no real points it gives only "misses" (O3) or two non-real, non-conjugate points (O6). The matrix is symmetric because common points do not depend on which object is the mirror; only the image $C \star L$ depends on the order. The P1/P2 rows have no matching columns, since points cannot be mirrors (Prop. 3.1).

Table 6: One exact example per leaf: input C, mirror L, κ, class of $C \star L$, and common points (3 decimals; "ideal" = a common point at infinity on $\mathcal{Q}$, i.e. ∞ or an isotropic direction; $^{\times 2}$ = double). [G1]

#	cell	O	input C	mirror L	κ	img	common points
1	M1×M1	1	$x^2+y^2-1=0$	$x^2+y^2-2x=0$	$\frac{1}{4}$	M1	$(0.5, 0.866)$; $(0.5, -0.866)$
2	M1×M1	$1^{\perp}$	$y=0$	$x=0$	0	M1	ideal; $(0,0)$
3	M1×M1	2	$x^2+y^2-2x=0$	$x=0$	1	M1	$(0,0)^{\times 2}$
4	M1×M1	3	$x^2+y^2-6x+8=0$	$x=0$	9	M1	$(0, -2.828i)$; $(0, 2.828i)$
5	M1×M2	3	$x^2+y^2-4=0$	$x^2+y^2+1=0$	$-\frac{9}{16}$	M1	ideal; ideal
6	M1×M2	$3^{\perp}$	$y=0$	$x^2+y^2+1=0$	0	M1	$(-i, 0)$; $(i, 0)$
7	M1×M3	1	$y=0$	$x+2iy=0$	$\frac{4}{3}$	M3	ideal; $(0,0)$
8	M1×M3	4	$x^2+y^2-2x=0$	$x+2iy=0$	$-\frac{1}{3}$	M3	$(2.667, 1.333i)$; $(0,0)$
9	M1×M3	5	$x^2+y^2-2x=0$	$x^2+y^2-2iy-2=0$	1	M3	ideal$^{\times 2}$
10	M1×M3	6	$x^2+y^2-6x+8=0$	$x+2iy=0$	-3	M3	$(6.309, 3.155i)$; $(1.691, 0.845i)$
11	M1×M3	$6^{\perp}$	$x^2+y^2-1=0$	$x+2iy=0$	0	M1	$(-1.155, -0.577i)$; $(1.155, 0.577i)$
12	M1×M4	2	$x=0$	$x-i=0$	1	M4	ideal$^{\times 2}$
13	M1×M4	4	$y=0$	$x+y-i=0$	$\frac{1}{2}$	M4	ideal; $(i, 0)$
14	M1×M4	$4^{\perp}$	$y=0$	$x-i=0$	0	M1	ideal; $(i, 0)$
15	M1×M4	6	$x^2+y^2-1=0$	$x-i=0$	-1	M3	$(i, 1.414)$; $(i, -1.414)$
16	M1×M5	3	$x^2+y^2-1=0$	$x^2+y^2+i=0$	$\frac{1}{2}$	M2	ideal; ideal

#	cell	O	input C	mirror L	κ	img	common points
17	M1×M5	6	$x^2+y^2-2x=0$	$x^2+y^2+i=0$	$-\frac{1}{4}i$	M4	$(-0.5i, 0.8-0.625i)$; $(-0.5i, -0.8+0.625i)$
18	M1×M5	$6^\perp$	$y=0$	$x^2+y^2+i=0$	0	M1	$(0.707-0.707i, 0)$; $(-0.707+0.707i, 0)$
19	M2×M1	3	$x^2+y^2+1=0$	$x^2+y^2-4=0$	$-\frac{9}{16}$	M2	ideal; ideal
20	M2×M1	$3^\perp$	$x^2+y^2+1=0$	$y=0$	0	M2	$(-i, 0)$; $(i, 0)$
21	M2×M2	3	$x^2+y^2+1=0$	$x^2+y^2-2x+2=0$	$\frac{9}{4}$	M2	$(0.5, -1.118i)$; $(0.5, 1.118i)$
22	M2×M3	5	$x^2+y^2+1=0$	$x^2+y^2-4ix-3=0$	1	M3	$(i, 0)^{\times 2}$
23	M2×M3	6	$x^2+y^2-2x+2=0$	$x+2iy=0$	$\frac{1}{3}$	M3	$(1.333-0.943i, 0.471+0.667i)$; $(1.333+0.943i, -0.471+0.667i)$
24	M2×M3	$6^\perp$	$x^2+y^2+1=0$	$x+2iy=0$	0	M2	$(-1.155i, 0.577)$; $(1.155i, -0.577)$
25	M2×M4	5	$x^2+y^2+1=0$	$x-i=0$	1	M3	$(i, 0)^{\times 2}$
26	M2×M4	6	$x^2+y^2+1=0$	$x+y-i=0$	$\frac{1}{2}$	M3	$(i, 0)$; $(0, i)$
27	M2×M5	3	$x^2+y^2+1=0$	$x^2+y^2+i=0$	$\frac{1}{2}$	M1	ideal; ideal
28	M2×M5	5	$3(x^2+y^2)+x+2=0$	$4(x^2+y^2)+5x-4iy+2=0$	1	M3	$(-2, 2i)^{\times 2}$
29	M2×M5	6	$x^2+y^2-2x+2=0$	$x^2+y^2+i=0$	$1-\frac{3}{4}i$	M3	$(1-0.5i, -0.866i)$; $(1-0.5i, 0.866i)$
30	M3×M1	1	$x+2iy=0$	$y=0$	$\frac{4}{3}$	M3	ideal; $(0, 0)$
31	M3×M1	4	$x+2iy=0$	$x^2+y^2-2x=0$	$-\frac{1}{3}$	M3	$(2.667, 1.333i)$; $(0, 0)$
32	M3×M1	5	$x^2+y^2-2iy-2=0$	$x^2+y^2-2x=0$	1	M3	ideal$^{\times 2}$
33	M3×M1	6	$x+2iy=0$	$x^2+y^2-6x+8=0$	-3	M3	$(6.309, 3.155i)$; $(1.691, 0.845i)$
34	M3×M1	$6^\perp$	$x+2iy=0$	$x^2+y^2-1=0$	0	M3	$(-1.155, -0.577i)$; $(1.155, 0.577i)$
35	M3×M2	5	$x^2+y^2-4ix-3=0$	$x^2+y^2+1=0$	1	M3	$(i, 0)^{\times 2}$
36	M3×M2	6	$x+2iy=0$	$x^2+y^2-2x+2=0$	$\frac{1}{3}$	M3	$(1.333-0.943i, 0.471+0.667i)$; $(1.333+0.943i, -0.471+0.667i)$
37	M3×M2	$6^\perp$	$x+2iy=0$	$x^2+y^2+1=0$	0	M3	$(-1.155i, 0.577)$; $(1.155i, -0.577)$
38	M3×M3	1	$i(x^2+y^2)-3ix+(-2-i)y-i=0$	$2(x^2+y^2)-6x+(-6-4i)y-2=0$	$\frac{128}{87}-\frac{88}{87}i$	M3	$(3.303, 0)$; $(-0.303, 0)$
39	M3×M3	$1^\perp$	$2ix-y=0$	$x+2iy=0$	0	M3	ideal; $(0, 0)$
40	M3×M3	2	$(1+4i)(x^2+y^2)+(1-i)x+3y=0$	$(2+i)(x^2+y^2)+(1-i)x+3y=0$	1	M3	$(0, 0)^{\times 2}$
41	M3×M3	4	$(-1+i)(x^2+y^2)+(1-i)x-2y=0$	$(2+i)(x^2+y^2)+(3+3i)x+(2+3i)y=0$	$-\frac{44}{185}+\frac{32}{185}i$	M3	$(0.224+0.006i, -1.094-0.924i)$; $(0, 0)$
42	M3×M3	$4^\perp$	$3i(x^2+y^2)+x+(1-i)y=0$	$(3+i)(x^2+y^2)-2x+(1+i)y=0$	0	M3	$(0.346+0.769i, 0.731-0.154i)$; $(0, 0)$
43	M3×M3	5	$x^2+y^2-4ix-3=0$	$x^2+y^2+(-2-2i)x+(-1+2i)=0$	1	M3	$(i, 0)^{\times 2}$
44	M3×M3	6	$x^2+y^2-2ix-2=0$	$x+2iy=0$	$\frac{1}{3}$	M3	$(-0.943+1.333i, -0.667-0.471i)$; $(0.943+1.333i, -0.667+0.471i)$
45	M3×M3	$6^\perp$	$(2+i)(x^2+y^2)+x+(-3-3i)y+2i=0$	$-\frac{5}{4}i(x^2+y^2)+3ix+(1+2i)y+(-1-i)=0$	0	M3	$(0.027-0.504i, 0.085+0.316i)$; $(0.251+0.15i, 1.412-0.037i)$
46	M3×M4	4	$x+2iy=0$	$x-i=0$	$-\frac{1}{3}$	M3	ideal; $(i, -0.5)$
47	M3×M4	5	$x^2+y^2-4ix-3=0$	$x-i=0$	1	M2	$(i, 0)^{\times 2}$
48	M3×M4	6	$x^2+y^2-4ix-3=0$	$x+y-i=0$	$\frac{1}{2}$	M3	$(i, 0)$; $(2i, -i)$
49	M3×M4	$6^\perp$	$x^2+y^2-2ix-2=0$	$x-i=0$	0	M3	$(i, -1)$; $(i, 1)$
50	M3×M5	5	$(1-i)(x^2+y^2)+(-2-2i)x+(1+6i)y+(-8-3i)=0$	$-i(x^2+y^2)-2ix+y-3i=0$	1	M3	$(1, -3i)^{\times 2}$
51	M3×M5	6	$x^2+y^2-2ix-2=0$	$x^2+y^2+i=0$	$1+\frac{3}{4}i$	M2	$(-0.5+i, 0.866)$; $(-0.5+i, -0.866)$
52	M3×M5	$6^\perp$	$x+2iy=0$	$x^2+y^2+i=0$	0	M3	$(0.816-0.816i, 0.408+0.408i)$; $(-0.816+0.816i, -0.408-0.408i)$

#	cell	O	input C	mirror L	κ	img	common points
53	M4×M1	2	$x-i=0$	$x=0$	1	M4	ideal$^{\times 2}$
54	M4×M1	4	$x+y-i=0$	$y=0$	$\frac{1}{2}$	M4	ideal; $(i,0)$
55	M4×M1	$4^{\perp}$	$x-i=0$	$y=0$	0	M4	ideal; $(i,0)$
56	M4×M1	6	$x-i=0$	$x^2{+}y^2-1=0$	-1	M4	$(i,1.414)$; $(i,-1.414)$
57	M4×M2	5	$x-i=0$	$x^2{+}y^2+1=0$	1	M4	$(i,0)^{\times 2}$
58	M4×M2	6	$x+y-i=0$	$x^2{+}y^2+1=0$	$\frac{1}{2}$	M4	$(i,0)$; $(0,i)$
59	M4×M3	4	$x-i=0$	$x+2iy=0$	$-\frac{1}{3}$	M3	ideal; $(i,-0.5)$
60	M4×M3	5	$x-i=0$	$x^2{+}y^2-4ix-3=0$	1	M3	$(i,0)^{\times 2}$
61	M4×M3	6	$x-i=0$	$x^2{+}y^2-2iy-2=0$	-1	M3	$(i,1.414+i)$; $(i,-1.414+i)$
62	M4×M3	$6^{\perp}$	$x-i=0$	$x^2{+}y^2-2ix-2=0$	0	M4	$(i,-1)$; $(i,1)$
63	M4×M4	2	$(2+2i)(x^2{+}y^2)+2x=0$	$(3-i)(x^2{+}y^2)+2x=0$	1	M4	$(0,0)^{\times 2}$
64	M4×M4	4	$x-i=0$	$x+y-i=0$	$\frac{1}{2}$	M1	ideal; $(i,0)$
65	M4×M4	$4^{\perp}$	$(\frac{3}{2}-i)(x^2{+}y^2)+(-6+4i)x-3y+(6-4i)=0$	$(2-2i)(x^2{+}y^2)+(-9+8i)x+(10-8i)=0$	0	M4	$(2,0)$; $(2.244+0.236i, 0.041+0.059i)$
66	M4×M4	5	$2x+3y+(4+13i)=0$	$i(x^2{+}y^2)+(-2+4i)x-3y+(-4+4i)=0$	1	M3	$(-2-2i,-3i)^{\times 2}$
67	M4×M4	6	$(4+2i)(x^2{+}y^2)+2x=0$	$(-2+i)(x^2{+}y^2)+(9-4i)x+(3-2i)y+(-11+5i)=0$	$\frac{3965}{2}-126i$	M3	$(0.943-0.317i, 0.405+1.126i)$; $(1.047+0.471i, 0.389-1.24i)$
68	M4×M4	$6^{\perp}$	$x-i=0$	$x^2{+}y^2-2ix=0$	0	M4	$(i,-i)$; (i,i)
69	M4×M5	5	$(2+3i)(x^2{+}y^2)-3x+(-4-12i)y+12i=0$	$(2+2i)(x^2{+}y^2)-x+(-6-10i)y+(2+13i)=0$	1	M3	$(-i,1+i)^{\times 2}$
70	M4×M5	6	$x-i=0$	$x^2{+}y^2+i=0$	$-i$	M1	$(i,1.099-0.455i)$; $(i,-1.099+0.455i)$
71	M4×M5	$6^{\perp}$	$i(x^2{+}y^2)+2x=0$	$(-2-3i)(x^2{+}y^2)+(3+3i)x+(1+2i)y+(3-3i)=0$	0	M4	$(-0.646+0.279i, 0.445-1.047i)$; $(-0.181+0.272i, -0.18+0.732i)$
72	M5×M1	3	$x^2{+}y^2+i=0$	$x^2{+}y^2-1=0$	$\frac{1}{2}$	M5	ideal; ideal
73	M5×M1	6	$x^2{+}y^2+i=0$	$x^2{+}y^2-2x=0$	$-\frac{1}{4}i$	M5	$(-0.5i, 0.8-0.625i)$; $(-0.5i, -0.8+0.625i)$
74	M5×M1	$6^{\perp}$	$x^2{+}y^2+i=0$	$y=0$	0	M5	$(0.707-0.707i, 0)$; $(-0.707+0.707i, 0)$
75	M5×M2	3	$x^2{+}y^2+i=0$	$x^2{+}y^2+1=0$	$\frac{1}{2}$	M5	ideal; ideal
76	M5×M2	5	$(1-\frac{1}{2}i)(x^2{+}y^2)-x+(-1+i)y+1=0$	$x^2{+}y^2-y+1=0$	1	M5	$(i,1)^{\times 2}$
77	M5×M2	6	$x^2{+}y^2+i=0$	$x^2{+}y^2-2x+2=0$	$1-\frac{3}{4}i$	M5	$(1-0.5i, -0.866i)$; $(1-0.5i, 0.866i)$
78	M5×M3	5	$x^2{+}y^2-ix+(-1-i)y+(2+i)=0$	$3(x^2{+}y^2)+(-8-5i)x+(-5+7i)y+(2+i)=0$	1	M3	$(2+i,1-2i)^{\times 2}$
79	M5×M3	6	$x^2{+}y^2+i=0$	$x^2{+}y^2-2ix-2=0$	$1+\frac{3}{4}i$	M3	$(-0.5+i, 0.866)$; $(-0.5+i, -0.866)$
80	M5×M3	$6^{\perp}$	$x^2{+}y^2+i=0$	$x+2iy=0$	0	M5	$(0.816-0.816i, 0.408+0.408i)$; $(-0.816+0.816i, -0.408-0.408i)$
81	M5×M4	5	$(-\frac{5}{4}+\frac{1}{4}i)(x^2{+}y^2)+2ix+(2-2i)y+(-1+3i)=0$	$-x^2{+}y^2+(-1+2i)x+(1-2i)y+4i=0$	1	M3	$(1+i,1+i)^{\times 2}$
82	M5×M4	6	$x^2{+}y^2+i=0$	$x-i=0$	$-i$	M3	$(i,1.099-0.455i)$; $(i,-1.099+0.455i)$
83	M5×M4	$6^{\perp}$	$(3-i)(x^2{+}y^2)+x+(1-2i)y+(2+i)=0$	$\frac{1}{2}i(x^2{+}y^2)-y=0$	0	M5	$(0.081-0.671i, 0.072-0.251i)$; $(-0.278+0.685i, 0.234-0.186i)$
84	M5×M5	3	$-x^2{+}y^2+3x+iy+(1-2i)=0$	$2(x^2{+}y^2)-6x+4iy+(-2-8i)=0$	$\frac{15931}{17524}+\frac{405}{8762}i$	M5	$(1.5-0.866i, 2)$; $(1.5+0.866i, 2)$
85	M5×M5	$3^{\perp}$	$x^2{+}y^2-i=0$	$x^2{+}y^2+i=0$	0	M5	ideal; ideal
86	M5×M5	5	$3(x^2{+}y^2)-2x+2i=0$	$(3+2i)(x^2{+}y^2)+2x-4iy+2i=0$	1	M3	$(i,1)^{\times 2}$
87	M5×M5	6	$x^2{+}y^2+i=0$	$x^2{+}y^2-ix+\frac{3}{4}=0$	$\frac{3}{8}+\frac{7}{64}i$	M5	$(-1-0.75i, 1.025-1.22i)$; $(-1-0.75i, -1.025+1.22i)$
88	M5×M5	$6^{\perp}$	$2(x^2{+}y^2)+(-1+i)x+(-1+i)y+(2+3i)=0$	$(-1+i)(x^2{+}y^2)-x-y+3=0$	0	M5	$(-0.512+2.742i, 3.012+0.258i)$; $(3.012+0.258i, -0.512+2.742i)$

#	cell	O	input C	mirror L	κ	img	common points
89	P1×M1	2	$\mathrm{pt}(0,0)$	$y=0$	–	P1	$(0,0)^{\times 2}$
90	P1×M1	3	$\mathrm{pt}(0,0)$	$x^2{+}y^2-1=0$	–	P1	ideal; ideal
91	P1×M2	3	$\mathrm{pt}(0,0)$	$x^2{+}y^2+1=0$	–	P1	ideal; ideal
92	P1×M3	2	$\mathrm{pt}(0,0)$	$x+2iy=0$	–	P1	$(0,0)^{\times 2}$
93	P1×M3	6	$\mathrm{pt}(2,0)$	$x+2iy=0$	–	P2	$(4,2i)$; $(1.333,0.667i)$
94	P1×M4	2	$\mathrm{pt}(0,0)$	$x^2{+}y^2-2ix=0$	–	P1	$(0,0)^{\times 2}$
95	P1×M4	6	$\mathrm{pt}(0,0)$	$x-i=0$	–	P2	$(i,-1)$; $(i,1)$
96	P1×M5	3	$\mathrm{pt}(0,0)$	$x^2{+}y^2+i=0$	–	P1	ideal; ideal
97	P1×M5	6	$\mathrm{pt}(2,0)$	$x^2{+}y^2+i=0$	–	P2	$(1-0.25i, 0.25-i)$; $(1-0.25i, -0.25+i)$
98	P2×M1	1	$\mathrm{pt}(i,0)$	$x=0$	–	P2	$(0,1)$; $(0,-1)$
99	P2×M1	4	$\mathrm{pt}(0,2i)$	$x^2{+}y^2-2x=0$	–	P2	$(2,0)$; $(-0.667,1.333i)$
100	P2×M1	5	$\mathrm{pt}(i,0)$	$y=0$	–	P2	$(i,0)^{\times 2}$
101	P2×M1	6	$\mathrm{pt}(i,0)$	$x^2{+}y^2-4=0$	–	P2	$(-1.5i, 2.5)$; $(-1.5i, -2.5)$
102	P2×M2	5	$\mathrm{pt}(i,0)$	$x^2{+}y^2+1=0$	–	P2	$(i,0)^{\times 2}$
103	P2×M2	6	$\mathrm{pt}(0,2i)$	$x^2{+}y^2+1=0$	–	P2	$(-0.75, 1.25i)$; $(0.75, 1.25i)$
104	P2×M3	1	$\mathrm{pt}(-i,2)$	$x^2{+}y^2{+}(3{+}i)x{-}4y{+}3=0$	–	P2	$(0,1)$; $(0,3)$
105	P2×M3	4	$\mathrm{pt}(-\frac{1}{2}-\frac{1}{2}i, -\frac{1}{2}+\frac{1}{2}i)$	$2(x^2{+}y^2)+2x+(-3+i)y=0$	–	P2	$(-0.862-0.655i, -0.345+0.138i)$; $(0,0)$
106	P2×M3	5	$\mathrm{pt}(i,0)$	$x^2{+}y^2-4ix-3=0$	–	P2	$(i,0)^{\times 2}$
107	P2×M3	6	$\mathrm{pt}(i,0)$	$x+2iy=0$	–	P2	$(2i,-1)$; $(0.667i,-0.333)$
108	P2×M4	4	$\mathrm{pt}(\frac{1}{2}-\frac{1}{2}i, -\frac{1}{2}-\frac{1}{2}i)$	$(3{+}i)(x^2{+}y^2){-}2x{+}y=0$	–	P2	$(0.733-0.467i, -0.533-0.267i)$; $(0,0)$
109	P2×M4	5	$\mathrm{pt}(i,0)$	$x-i=0$	–	P2	$(i,0)^{\times 2}$
110	P2×M4	6	$\mathrm{pt}(0,2i)$	$x-i=0$	–	P2	$(i,1+2i)$; $(i,-1+2i)$
111	P2×M5	5	$\mathrm{pt}(-2,i)$	$\frac{1}{3}(x^2{+}y^2)+(-3+3i)y+(2+3i)=0$	–	P2	$(-2,i)^{\times 2}$
112	P2×M5	6	$\mathrm{pt}(i,0)$	$x^2{+}y^2+i=0$	–	P1	$(-0.5+0.5i, 0.5-0.5i)$; $(-0.5+0.5i, -0.5+0.5i)$

7 Counting the collapse

level	count	what is identified
raw ordered (input, operator) types	2025	45×45
of which operator is a point	405	45×9: one degenerate rule (Prop. 3.1); isotropic lines ($m = \pm i$ inside line types `I*`) obey the same rule
with a line/circle operator	1620	45×36
real class pairs	49	7×7 (Thm. 2.2, Prop. 2.4)
valid cells	35	7 inputs × 5 mirror classes; 14 point-mirror pairs → 1 rule
leaves (cell, outcome)	88	Theorem 6.1
+ orthogonal sub-leaves	24	in 20 of the 25 M×M cells
complex orbit types of pairs	6	Prop. 4.1 (plus the value of κ)
Euclidean real relations	18 → 4	κ-classes (Sec. 5)
real point vs. real circle/line	8 → 2	on / off

Over $\mathbb{C}$, the 88 leaves collapse to just "two distinct" or "double" common points (plus coincidence and incidence for points). The six-way split O1–O6 is entirely an effect of the real structure.

8 Surprises

1. **All non-real points are one point.** $(i,0)$, $(1+i, 2-3i)$ and even the isotropic lines are the same P2 object up to real Möbius maps. Each non-real point carries a real "shadow", its

Laguerre pair, and inverting it in a real circle through that pair returns its conjugate.

2. **The parameter types barely matter.** A complex centre with a *real* radius always gives a circle with two real points (M3). A line is never of the no-real-point type M5, because it passes through ∞. Most circle types are decided by the sign of $\Delta = 4|w|^2(\operatorname{Re}\rho + |w|^2) - (\operatorname{Im}\rho)^2$, not by which parameters are imaginary.

3. **Real tangency needs equal τ, and tangency to a real circle at a non-real point forces M3.** Consequently a real circle and an M5 (e.g. $x^2 + y^2 = i$) can only miss (O3) or meet in two unrelated non-real points (O6): no tangency of any kind.

4. **Imaginary-radius circles are rigid.** Two of them always have $\kappa > 1$: never tangent, never orthogonal. Orthogonality fails in exactly five cells, all involving M2. A real circle and an imaginary-radius one are swapped by a non-real mirror only via $\tau = 0$ mirrors such as $x^2 + y^2 = \pm i$.

5. **A point as a mirror is a limit with no value**: the limit depends on how the mirror circle shrinks.

9 Verification map and honesty notes

`verify/classification_verify.py` ran 305 checks with 0 failures (log: `verify/classification_verify_log.txt`). The tags are S1 (raw types, 57), S2/S3 (single objects), M1–M3 (mirrors), D1 (degenerate mirror), L0–L3 (lemmas), K1/K2 (κ and signs), E1/E2 (Euclidean tables), G0–G2 (grid), and I1 (image classes).

- *Proved in the text:* Theorem 2.2, Propositions 2.4, 3.1, 3.3, Lemmas 4.2–4.5, and the exclusions of Theorem 6.1. *Quoted:* Witt's theorem (Props. 2.1, 4.1).
- *Exact, by example:* every realised leaf (Table 6).
- *Observed only:* the non-real-mirror columns of Table 2 (lower bounds), and the sampled κ ranges.
- The search (random, Laguerre, tangent, pencil and orthogonal recipes, plus symmetric swaps) never produced an outcome outside Table 5. That is consistent with, but no substitute for, the proofs.
- Changes made during review are recorded in `classification/classification_review_log.txt`.

References

[1] E. Witt, Theorie der quadratischen Formen in beliebigen Körpern, *J. reine angew. Math.* 176 (1937), 31–44.