Sixteen worked examples of $C \star L$

Reflecting generalized circles in $\mathbb{P}^3(\mathbb{C})$: real, complex and mixed

Setting. A generalized circle is $C = [a : b : c : d] \in \mathbb{P}^3(\mathbb{C})$, the curve $a(x^2+y^2)+bx+cy+d = 0$; lines have $a = 0$. With

$$Q(C) = b^2 + c^2 - 4ad, \qquad \langle C, C' \rangle = bb' + cc' - 2(ad' + a'd),$$

$$C \star L = C - \frac{2\langle C, L\rangle}{Q(L)}\, L \quad (Q(L) \neq 0), \qquad \kappa(C, L) = \frac{\langle C, L\rangle^2}{Q(C)\, Q(L)},$$

everything is bilinear over $\mathbb{C}$ (no complex conjugates). For a real circle $Q/a^2 = 4r^2$. A point $p = (x_0, y_0) \in \mathbb{C}^2$ is the point-circle $\mathrm{pt}(p) = [1 : -2x_0 : -2y_0 : x_0^2 + y_0^2]$, with $Q = 0$, and $\langle \mathrm{pt}(p), C\rangle = -2\, C(p)$; $\infty = [0 : 0 : 0 : 1]$.

Facts used. (i) $C \star L$ is again a generalized circle, $Q(C \star L) = Q(C)$, and $(C \star L) \star L = C$; real mirrors act as reflection in a line or inversion in a circle. (ii) $C \star L = C - kL$ lies in the pencil of C and L; if $k \neq 0$, i.e. $\langle C, L\rangle \neq 0$, then $C \cap (C \star L) = C \cap L$. (iii) The common points of C and L on the Möbius quadric $\{Q = 0\}$ are the null vectors of $\mathrm{span}(C, L)^{\perp}$: two, or one double point when $\kappa = 1$ (tangency). (iv) κ is symmetric and unchanged by every reflection; for two real lines or for a real crossing it is $\cos^2$ of the angle.

Caveats. $Q(L) = 0$ (point-circles, isotropic lines $b^2 + c^2 = 0$): L cannot be a mirror, and κ is undefined if either Q vanishes. $\langle C, L\rangle = 0$: $C \star L = C$, and fact (ii) gives no information. "Two points" counts multiplicity and is about the Möbius quadric: a line ($a = 0$) meets another line at ∞; concentric circles and isotropic lines can lose finite points, which then sit on the quadric over ∞ (Examples 5, 10). "Real picture" means we draw only real points, or the real trace $\mathrm{Re}\, C \cap \mathrm{Im}\, C$ of a complex circle, and say so.

Verification. Every number below is computed exactly with SymPy and asserted in `examples16.py`. That includes: the image, κ, the common points, involution, invariance of Q and κ, $C \cap (C \star L) = C \cap L$ when $\langle C, L\rangle \neq 0$, incidence of the Möbius points, a floating-point re-check under random complex rescaling, and each example's special claims. Log: `examples16_verify_log.txt` (247 checks, no failures).

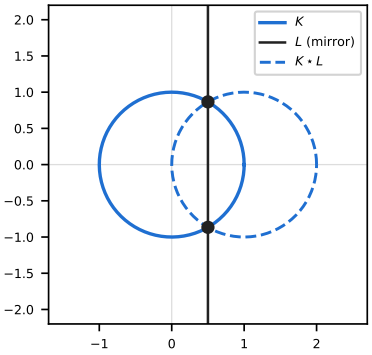
#	example	inputs	κ	finite common points
1	Fold the unit circle along $x = \frac{1}{2}$	real	$\frac{1}{4}$	two real
2	The line that misses: $x = 2$	real input, hidden output	4	conjugate pair
3	Just touching: $x + y = \sqrt{2}$	real	1	one double point (real)
4	Orthogonal circles: the mirror that does nothing	real	0	two real
5	Concentric circles meet only at infinity	real	$\frac{25}{16}$	no finite point
6	Two lines at 60°	real	$\frac{1}{4}$	one finite point (and ∞)
7	Invert a line: $x = 2$ in the unit circle	real input, hidden output	4	conjugate pair
8	A circle with complex centre: $(x - i)^2 + y^2 = 1$	complex	-1	two real
9	The circle of imaginary radius $x^2 + y^2 = -1$	mixed	$-\frac{1}{4}$	conjugate pair
10	Isotropic lines: $y = ix$, $y = ix + 1$, and $y = 2ix$	complex	0	two complex, not conjugate
11	A point as input: the point-circle at $(1, 1)$	real, degenerate	undefined ($Q(\cdot) = 0$)	conjugate pair
12	A complex mirror makes a real circle non-real	mixed	$\frac{9}{4}$	two complex, not conjugate
13	Two mirrors make a rotation of order 2, 3, 4, ∞, or a complex angle	real and complex	4	one finite point (and ∞)
14	The kei law, checked by hand	real	$\frac{4}{5}$	two real
15	Tangency with complex coefficients	complex	1	one double point (complex)
16	Find the mirror: swap two disjoint circles	real input, imaginary answer	$\frac{121}{16}$	conjugate pair

Example 1. Fold the unit circle along $x = \frac{1}{2}$

Exercise. Reflect the unit circle K in the line $L:\ x = \frac{1}{2}$. Find $K \star L$, the common points and κ.

Solution. Inputs. $K = [1 : 0 : 0 : -1]$, i.e. $x^2 + y^2 = 1$; $L = [0 : 1 : 0 : -\frac{1}{2}]$, i.e. $x = \frac{1}{2}$. **Numbers.** $Q(K) = 4$, $Q(L) = 1$, $\langle K, L\rangle = 1$, $\kappa = \frac{1}{4}$. **Image.** $K \star L = K - \frac{2\langle K,L\rangle}{Q(L)} L \sim [1 : -2 : 0 : 0]$, i.e. $x^2 - 2x + y^2 = 0$. **Common points** (finite, of K and L): $\left(\frac{1}{2},\ -\frac{\sqrt{3}}{2}\right)$, $\left(\frac{1}{2},\ \frac{\sqrt{3}}{2}\right)$ — two real. The image is the unit circle translated to centre $(1, 0)$, as a fold along $x = \frac{1}{2}$ should give. The real crossing angle φ satisfies $\kappa = \cos^2 \varphi$: here $\varphi = 60°$.

Why it is fun. For a real secant, $\kappa = \cos^2$(crossing angle); for our sweep $x = t$, $\kappa = t^2$.



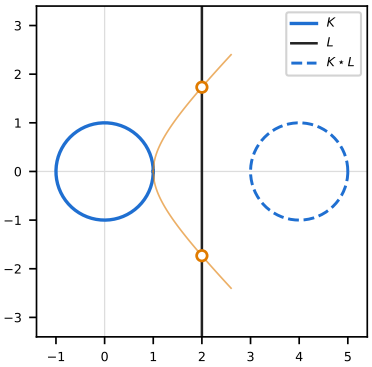
Real picture: fold along $x = \frac{1}{2}$.

Example 2. The line that misses: $x = 2$

Exercise. Reflect K in $L:\ x = 2$, which misses it. Where do K and L meet?

Solution. Inputs. $K = [1 : 0 : 0 : -1]$, i.e. $x^2 + y^2 = 1$; $L = [0 : 1 : 0 : -2]$, i.e. $x = 2$. **Numbers.** $Q(K) = 4$, $Q(L) = 1$, $\langle K, L\rangle = 4$, $\kappa = 4$. **Image.** $K \star L = K - \frac{2\langle K,L\rangle}{Q(L)} L \sim [1 : -8 : 0 : 15]$, i.e. $x^2 - 8x + y^2 + 15 = 0$. **Common points** (finite, of K and L): $\left(2,\ -\sqrt{3}i\right)$, $\left(2,\ \sqrt{3}i\right)$ — conjugate pair. The image $(x-4)^2 + y^2 = 1$ is real, but the common points are the conjugate pair $(2, \pm i\sqrt{3})$. The image passes through them too: $4 - 16 + 15 + (i\sqrt{3})^2 = 0$. In the Short's convention (perpendicular sweep, points at the foot) we draw them as $(2, \pm\sqrt{3})$, a point of the hyperbola $t^2 - s^2 = 1$.

Why it is fun. $\kappa = 4 > 1$: real data, hidden points, and the algebra does not notice.



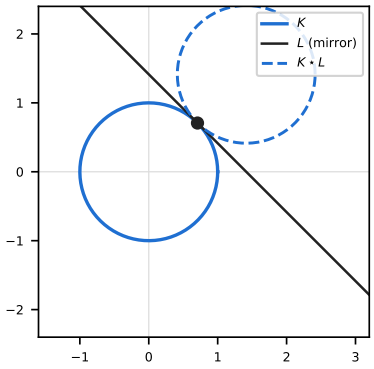
Real picture; rings: the slice drawing $(2, \pm\sqrt{3})$ of the hidden pair $(2, \pm i\sqrt{3})$.

Example 3. Just touching: $x + y = \sqrt{2}$

Exercise. Show that $L:\ x + y = \sqrt{2}$ is tangent to K using only κ, then find $K \star L$.

Solution. Inputs. $K = [1 : 0 : 0 : -1]$, i.e. $x^2 + y^2 = 1$; $L = [0 : 1 : 1 : -\sqrt{2}]$, i.e. $x + y = \sqrt{2}$. **Numbers.** $Q(K) = 4$, $Q(L) = 2$, $\langle K, L\rangle = 2\sqrt{2}$, $\kappa = 1$. **Image.** $K \star L = K - \frac{2\langle K,L\rangle}{Q(L)} L \sim [1 : -2\sqrt{2} : -2\sqrt{2} : 3]$, i.e. $x^2 - 2\sqrt{2}x + y^2 - 2\sqrt{2}y + 3 = 0$. **Common points** (finite, of K and L): $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ — one double point (real). $\kappa = 1$, so the two common points merge into one double point. The image circle, centre $(\sqrt{2}, \sqrt{2})$ and radius 1, touches K at the same point.

Why it is fun. Tangency is a single equation, $\langle C, L\rangle^2 = Q(C)Q(L)$; no square roots are needed.



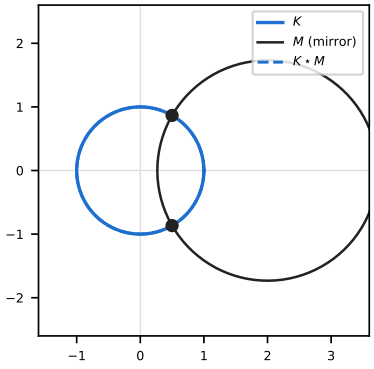
Real picture: one double point.

Example 4. Orthogonal circles: the mirror that does nothing

Exercise. Compute $K \star M$ for $M:\ (x-2)^2 + y^2 = 3$. Do K and M meet?

Solution. Inputs. $K = [1 : 0 : 0 : -1]$, i.e. $x^2 + y^2 = 1$; $M = [1 : -4 : 0 : 1]$, i.e. $(x-2)^2 + y^2 = 3$. **Numbers.** $Q(K) = 4$, $Q(M) = 12$, $\langle K, M\rangle = 0$, $\kappa = 0$. **Image.** $K \star M = K - \frac{2\langle K,M\rangle}{Q(M)} M \sim [1 : 0 : 0 : -1]$, i.e. $x^2 + y^2 - 1 = 0$. **Common points** (finite, of K and M): $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ — two real. $\langle K, M\rangle = 0$, so $K \star M = K$ (and $M \star K = M$). They do meet, at $(\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$, at right angles $(1 + 3 = 2^2)$. Here $C \cap (C \star L) = C$, so the image tells us nothing about the common points: this is the one exception to "$C \star L$ meets C where L does".

Why it is fun. $\kappa = 0$ means orthogonal. The mirror fixes the circle without fixing its points: inversion in M swaps the two arcs of K.



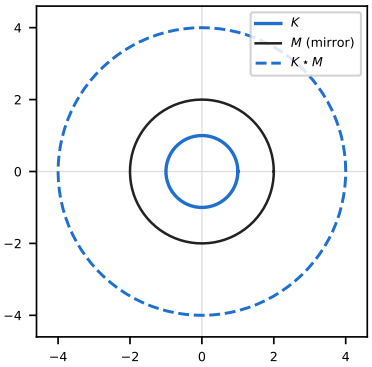
Real picture: orthogonal circles; the image is K itself.

Example 5. Concentric circles meet only at infinity

Exercise. Invert K in $M: x^2+y^2=4$. Where do two concentric circles meet?

Solution. Inputs. $K=[1:0:0:-1]$, i.e. $x^2+y^2=1$; $M=[1:0:0:-4]$, i.e. $x^2+y^2=4$. **Numbers.** $Q(K)=4$, $Q(M)=16$, $\langle K,M\rangle=10$, $\kappa=\frac{25}{16}$. **Image.** $K\star M=K-\frac{2\langle K,M\rangle}{Q(M)}M\sim[1:0:0:-16]$, i.e. $x^2+y^2-16=0$. **Common points** (finite, of K and M): none in $\mathbb{C}^2$ — no finite point. $K\star M$ is $x^2+y^2=16$, since inversion sends radius 1 to 4/1. There is no finite common point: subtracting the equations gives $3=0$. On the Möbius quadric the two common points are $[0:1:\pm i:0]$, i.e. the isotropic lines $x\pm iy=0$ through the common centre, which are null vectors "over ∞". The value $\kappa=\left(\frac{r_1^2+r_2^2-d^2}{2r_1r_2}\right)^2=(\frac{5}{4})^2$ is the squared inversive distance.

Why it is fun. $\kappa>1$ yet nothing hidden in $\mathbb{C}^2$: all the intersection sits at infinity.



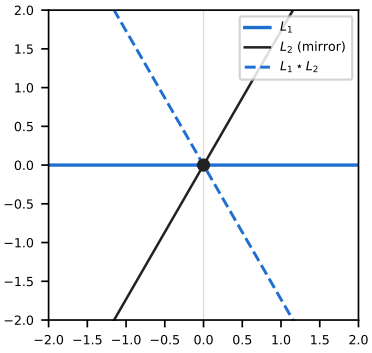
Real picture: concentric, no finite common point.

Example 6. Two lines at 60°

Exercise. Reflect the x-axis in the line $y=\sqrt{3}\,x$. Compute κ and the common points.

Solution. Inputs. $L_1=[0:0:1:0]$, i.e. $y=0$; $L_2=[0:\sqrt{3}:-1:0]$, i.e. $y=\sqrt{3}\,x$. **Numbers.** $Q(L_1)=1$, $Q(L_2)=4$, $\langle L_1,L_2\rangle=-1$, $\kappa=\frac{1}{4}$. **Image.** $L_1\star L_2=L_1-\frac{2\langle L_1,L_2\rangle}{Q(L_2)}L_2\sim[0:1:\frac{\sqrt{3}}{3}:0]$, i.e. $x+\frac{\sqrt{3}y}{3}=0$. **Common points** (finite, of L_1 and L_2): $(0,\ 0)$ — one finite point (and ∞). $\kappa=\frac{1}{4}=\cos^2 60°$. The image $y=-\sqrt{3}\,x$ makes $120°$ with the x-axis. The finite common point is the origin; the second is $\infty=[0:0:0:1]$, shared by all lines $(a=0)$.

Why it is fun. For two lines, κ is literally $\cos^2$ of the angle (see Example 13 for what composing does).



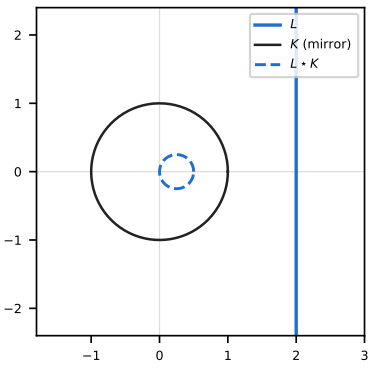
Real picture: the image line is at 120°.

Example 7. Invert a line: $x = 2$ in the unit circle

Exercise. Compute $L \star K$ for $L:\ x = 2$ and compare with Example 2, where the roles were swapped.

Solution. Inputs. $L = [0 : 1 : 0 : -2]$, i.e. $x = 2$; $K = [1 : 0 : 0 : -1]$, i.e. $x^2 + y^2 = 1$. **Numbers.** $Q(L) = 1$, $Q(K) = 4$, $\langle L, K\rangle = 4$, $\kappa = 4$. **Image.** $L \star K = L - \frac{2\langle L,K\rangle}{Q(K)} K \sim [1 : -\frac{1}{2} : 0 : 0]$, i.e. $x^2 - \frac{x}{2} + y^2 = 0$. **Common points** (finite, of L and K): $\big(2,\ -\sqrt{3}i\big)$, $\big(2,\ \sqrt{3}i\big)$ — conjugate pair. $L \star K$ is the circle $x^2 + y^2 = \frac{x}{2}$, centre $(\frac{1}{4}, 0)$, radius $\frac{1}{4}$, through the origin, which is the image of ∞. Like $K \star L$ in Example 2, it passes through the hidden points $(2, \pm i\sqrt{3})$. The value $\kappa = 4$ is the same, because κ is symmetric.

Why it is fun. $\star$ is not commutative: $K \star L \neq L \star K$, yet both live in the pencil spanned by K and L.



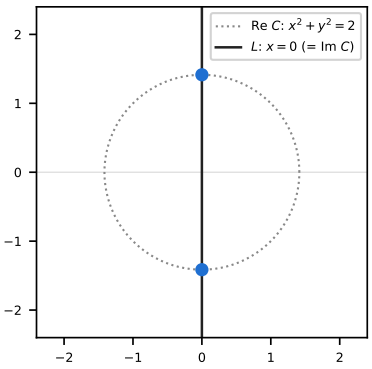
Real picture: the common points $(2, \pm i\sqrt{3})$ are not drawn.

Example 8. A circle with complex centre: $(x - i)^2 + y^2 = 1$

Exercise. Let $C:\ (x - i)^2 + y^2 = 1$. Reflect it in the y-axis. Then meet C with the x-axis.

Solution. Inputs. $C = [1 : -2i : 0 : -2]$, i.e. $(x-i)^2+y^2 = 1$; $L = [0 : 1 : 0 : 0]$, i.e. $x = 0$. **Numbers.** $Q(C) = 4$, $Q(L) = 1$, $\langle C, L\rangle = -2i$, $\kappa = -1$. **Image.** $C \star L = C - \frac{2\langle C,L\rangle}{Q(L)} L \sim [1 : 2i : 0 : -2]$, i.e. $x^2 + 2ix + y^2 - 2 = 0$. **Common points** (finite, of C and L): $\big(0,\ -\sqrt{2}\big)$, $\big(0,\ \sqrt{2}\big)$ — two real. $Q(C) = 4$, a "radius" of 1. $\kappa = -1 < 0$, which is impossible for a real circle and a real line. The reflection moves the centre from i to $-i$: $C \star L$ is $(x + i)^2 + y^2 = 1$. With the y-axis C meets in the real points $(0, \pm\sqrt{2})$; these are the whole real trace of C, where $\mathrm{Re} = x^2 + y^2 - 2$ and $\mathrm{Im} = -2x$ vanish together. With the x-axis $M' = [0 : 0 : 1 : 0]$: $\langle C, M'\rangle = 0$, so $C \star M' = C$, and the common points are $(1 + i, 0)$ and $(-1 + i, 0)$, complex and *not* conjugate.

Why it is fun. A complex circle can cross a real line in two real points, and in non-conjugate complex points.



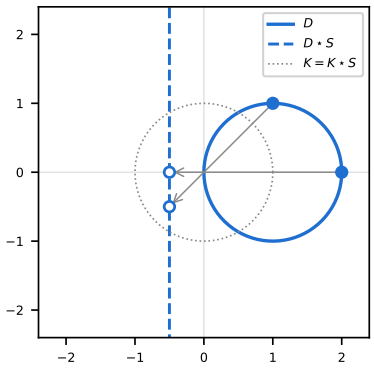
Only the real trace of C is drawn: $\mathrm{Re}\, C$ (grey circle) and $\mathrm{Im}\, C$ (the line $x = 0$, which is also the mirror) meet at $(0, \pm\sqrt{2})$.

Example 9. The circle of imaginary radius $x^2 + y^2 = -1$

Exercise. $S:\ x^2+y^2=-1$ has no real points. Use it as a mirror for $D:\ (x-1)^2+y^2=1$, and show $K \star S = K$ for the unit circle.

Solution. Inputs. $D = [1 : -2 : 0 : 0]$, i.e. $(x-1)^2 + y^2 = 1$; $S = [1 : 0 : 0 : 1]$, i.e. $x^2 + y^2 = -1$. **Numbers.** $Q(D) = 4$, $Q(S) = -4$, $\langle D, S\rangle = -2$, $\kappa = -\frac{1}{4}$. **Image.** $D \star S = D - \frac{2\langle D,S\rangle}{Q(S)} S \sim [0 : 1 : 0 : \frac{1}{2}]$, i.e. $x + \frac{1}{2} = 0$. **Common points** (finite, of D and S): $\left(-\frac{1}{2},\ -\frac{\sqrt{5}i}{2}\right)$, $\left(-\frac{1}{2},\ \frac{\sqrt{5}i}{2}\right)$ — conjugate pair. $Q(S) = -4 < 0$. $D \star S$ is the real line $x = -\frac{1}{2}$. In general, reflection in S sends the point p to $-p/|p|^2$ (inversion composed with the half-turn), a real map with no real fixed points. $\langle K, S\rangle = 0$, so $K \star S = K$: the unit circle is "orthogonal" to S (the map swaps antipodes of K). D and S meet in the conjugate pair $(-\frac{1}{2}, \pm\frac{\sqrt{5}}{2}i)$.

Why it is fun. An invisible mirror acts visibly: it maps real circles to real circles, and $\kappa = -\frac{1}{4} < 0$ betrays it.



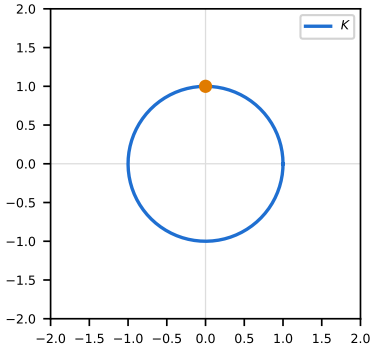
S has no real points (not drawn). Reflection in S is $p \mapsto -p/|p|^2$: D goes to the line $x = -\frac{1}{2}$.

Example 10. Isotropic lines: $y = ix$, $y = ix + 1$, and $y = 2ix$

Exercise. Which of $y = ix$, $y = ix + 1$, $y = 2ix$ can be a mirror? Meet each with K.

Solution. Inputs. $K = [1 : 0 : 0 : -1]$, i.e. $x^2 + y^2 = 1$; $L = [0 : 2i : -1 : 0]$, i.e. $y = 2ix$. **Numbers.** $Q(K) = 4$, $Q(L) = -3$, $\langle K, L\rangle = 0$, $\kappa = 0$. **Image.** $K \star L = K - \frac{2\langle K,L\rangle}{Q(L)} L \sim [1 : 0 : 0 : -1]$, i.e. $x^2 + y^2 - 1 = 0$. **Common points** (finite, of K and L): $\left(-\frac{\sqrt{3}i}{3},\ \frac{2\sqrt{3}}{3}\right)$, $\left(\frac{\sqrt{3}i}{3},\ -\frac{2\sqrt{3}}{3}\right)$ — two complex, not conjugate. $y = ix$ is $[0 : i : -1 : 0] \sim [0 : 1 : i : 0]$ and $y = ix + 1$ is $[0 : i : -1 : 1]$; both have $Q = i^2 + (-1)^2 = 0$, so neither can be a mirror. On $y = ix$ we have $x^2 + y^2 \equiv 0$, so K gives $-1 = 0$: no finite point. On the Möbius quadric the line $[0 : 1 : i : 0]$ is *itself* a null vector with $\langle K, \cdot\rangle = 0$, the double common point (tangency "at ∞"). $y = ix + 1$ meets K in the single finite point $(0, 1)$, which is also its only real point; the other common point is again $[0 : 1 : i : 0]$. $y = 2ix$ has $Q = -3 \neq 0$: a fine mirror. It passes through the centre, so $\langle K, L\rangle = 0$ and $K \star L = K$, with common points $\pm(\frac{i}{\sqrt{3}}, -\frac{2}{\sqrt{3}})$: each has one real and one imaginary coordinate, and they are not conjugate.

Why it is fun. Only the slopes $\pm i$ are forbidden. With them the "always two finite points" rule drops to one or zero.



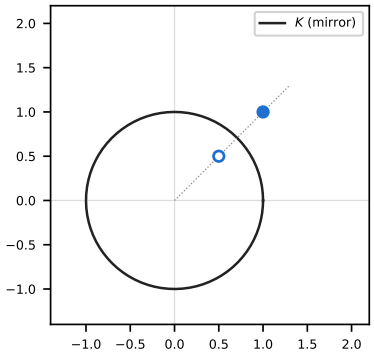
The non-real line $y = ix + 1$ has one real point, $(0, 1)$, which is also its only finite common point with K.

Example 11. A point as input: the point-circle at $(1, 1)$

Exercise. Treat the point $(1, 1)$ as the circle P of radius 0. Compute $P \star K$. What are κ and $P \cap K$?

Solution. Inputs. $P = [1 : -2 : -2 : 2]$, i.e. $(x-1)^2+(y-1)^2 = 0$; $K = [1 : 0 : 0 : -1]$, i.e. $x^2+y^2 = 1$. **Numbers.** $Q(P) = 0$, $Q(K) = 4$, $\langle P, K\rangle = -2$, $\kappa =$ undefined ($Q(\cdot) = 0$). **Image.** $P \star K = P - \frac{2\langle P,K\rangle}{Q(K)}K \sim [1 : -1 : -1 : \frac{1}{2}]$, i.e. $x^2 - x + y^2 - y + \frac{1}{2} = 0$. **Common points** (finite, of P and K): $\left(\frac{3}{4} - \frac{i}{4},\ \frac{3}{4} + \frac{i}{4}\right)$, $\left(\frac{3}{4} + \frac{i}{4},\ \frac{3}{4} - \frac{i}{4}\right)$ — conjugate pair. $Q(P) = 0$, so $P \star K$ is again a point-circle (since Q is preserved): the point $(\frac{1}{2}, \frac{1}{2})$, the classical inverse of $(1, 1)$. $K \star P$ is undefined and κ is undefined ($Q(P)Q(K) = 0$); only $\langle P, K\rangle = -2K(1, 1) = -2$, minus twice the power, is meaningful. P is the pair of isotropic lines through $(1, 1)$, and it meets K in the conjugate pair $(\frac{3}{4} \mp \frac{i}{4}, \frac{3}{4} \pm \frac{i}{4})$: the two points of K at zero (complex) distance from $(1, 1)$.

Why it is fun. Points are circles of $Q = 0$: inputs yes, mirrors no.



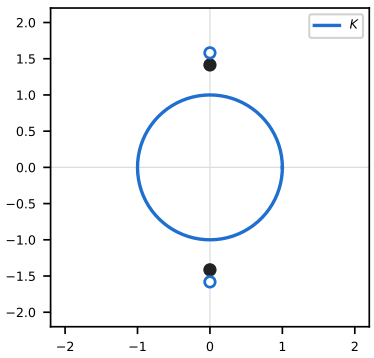
Real picture: the point $(1, 1)$ and its inverse $(\frac{1}{2}, \frac{1}{2})$; the common points are complex.

Example 12. A complex mirror makes a real circle non-real

Exercise. Reflect the real unit circle in $M:\ (x - i)^2 + y^2 = 1$. Is the image real? Where is it real?

Solution. Inputs. $K = [1 : 0 : 0 : -1]$, i.e. $x^2 + y^2 = 1$; $M = [1 : -2i : 0 : -2]$, i.e. $(x - i)^2 + y^2 = 1$. **Numbers.** $Q(K) = 4$, $Q(M) = 4$, $\langle K, M\rangle = 6$, $\kappa = \frac{9}{4}$. **Image.** $K \star M = K - \frac{2\langle K,M\rangle}{Q(M)}M \sim [1 : -3i : 0 : -\frac{5}{2}]$, i.e. $x^2 - 3ix + y^2 - \frac{5}{2} = 0$. **Common points** (finite, of K and M): $\left(\frac{i}{2},\ -\frac{\sqrt{5}}{2}\right)$, $\left(\frac{i}{2},\ \frac{\sqrt{5}}{2}\right)$ — two complex, not conjugate. $K \star M$ is $x^2 + y^2 - 3ix - \frac{5}{2} = 0$, i.e. $(x - \frac{3i}{2})^2 + y^2 = \frac{1}{4}$: complex centre, "radius" $\frac{1}{2}$. Its real trace ($x = 0$ and $x^2 + y^2 = \frac{5}{2}$) is the two points $(0, \pm\sqrt{5/2})$. K and M meet at $(\frac{i}{2}, \pm\frac{\sqrt{5}}{2})$: mixed, with imaginary x and real y.

Why it is fun. Realness is not preserved: a complex reflection takes a real circle off the real plane, leaving just two real points behind.



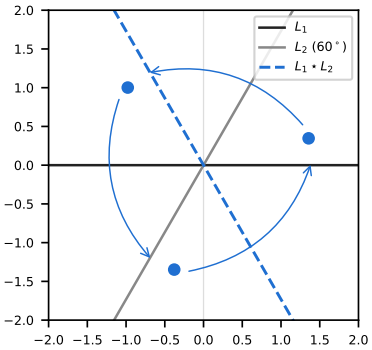
Real picture: K, plus the real traces of M (dots $(0, \pm\sqrt{2})$) and of $K \star M$ (rings $(0, \pm\sqrt{5/2})$).

Example 13. Two mirrors make a rotation of order 2, 3, 4, ∞, or a complex angle

Exercise. For lines through 0 at angle θ, $s_{L_2} \circ s_{L_1}$ is a rotation by 2θ. Tabulate κ and the order for $\theta = 90°, 60°, 45°$ and for parallel lines. Then take $L_2 : \; y = \frac{\sqrt{3}}{2} i\, x$.

Solution. Inputs. $L_1 = [0 : 0 : 1 : 0]$, i.e. $y = 0$; $L_2 = [0 : \frac{\sqrt{3}i}{2} : -1 : 0]$, i.e. $y = \frac{\sqrt{3}}{2} i\, x$. **Numbers.** $Q(L_1) = 1$, $Q(L_2) = \frac{1}{4}$, $\langle L_1, L_2 \rangle = -1$, $\kappa = 4$. **Image.** $L_1 \star L_2 = L_1 - \frac{2\langle L_1, L_2 \rangle}{Q(L_2)} L_2 \sim [0 : 1 : \frac{7\sqrt{3}i}{12} : 0]$, i.e. $x + \frac{7\sqrt{3}iy}{12} = 0$. **Common points** (finite, of L_1 and L_2): $(0,\ 0)$ — one finite point (and ∞). Real cases: $\theta = 90°$ gives $\kappa = 0$ and a half-turn (order 2); $60°$ gives $\kappa = \frac{1}{4}$ (order 3); $45°$ gives $\kappa = \frac{1}{2}$ (order 4). The parallel lines $x = 0$, $x = 1$ give $\kappa = 1$ (tangent at ∞) and the translation by 2 (infinite order). The $4{\times}4$ matrices satisfy $(R_2 R_1)^n = I$ for $n = 2, 3, 4$. Complex case: $\kappa = 4$, so "$\cos\theta = 2$". In isotropic coordinates $u = x + iy$, $v = x - iy$ the composition is $u \mapsto \lambda u$, $v \mapsto v/\lambda$ with $\lambda = (2 + \sqrt{3})^{\pm 2} = 7 \mp 4\sqrt{3}$: a boost, of infinite order. It sends real points (where $v = \bar{u}$) to non-real ones. $L_1 \star L_2$ is the line $x + \frac{7\sqrt{3}}{12} i\, y = 0$, whose slope $\frac{4\sqrt{3}}{7} i = \tan 2\theta$.

Why it is fun. $\kappa = \cos^2\theta$ keeps working when the angle is complex, and a rotation turns into a squeeze.



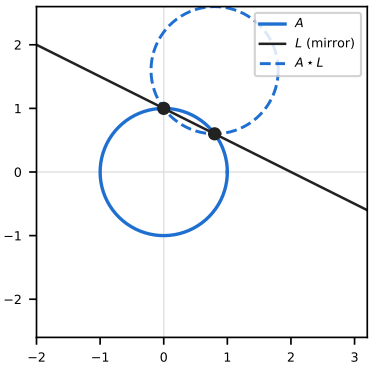
Real case 60°: $s_{L_2} \circ s_{L_1}$ rotates by 120° (order 3). The complex case has no real picture.

Example 14. The kei law, checked by hand

Exercise. With $A = K$, $B : \; x = \frac{1}{2}$, $L : \; x + 2y = 2$, verify $A \star A \sim A$, $(A \star L) \star L = A$, and $(A \star B) \star L = (A \star L) \star (B \star L)$. Is $\star$ associative?

Solution. Inputs. $A = [1 : 0 : 0 : -1]$, i.e. $x^2 + y^2 = 1$; $L = [0 : 1 : 2 : -2]$, i.e. $x + 2y = 2$. **Numbers.** $Q(A) = 4$, $Q(L) = 5$, $\langle A, L \rangle = 4$, $\kappa = \frac{4}{5}$. **Image.** $A \star L = A - \frac{2\langle A, L \rangle}{Q(L)} L \sim [1 : -\frac{8}{5} : -\frac{16}{5} : \frac{11}{5}]$, i.e. $x^2 - \frac{8x}{5} + y^2 - \frac{16y}{5} + \frac{11}{5} = 0$. **Common points** (finite, of A and L): $(0,\ 1)$, $\left(\frac{4}{5},\ \frac{3}{5}\right)$ — two real. $A \star A = -A \sim A$. $A \star B \sim [1 : -2 : 0 : 0]$, $A \star L \sim [1 : -\frac{8}{5} : -\frac{16}{5} : \frac{11}{5}]$, $B \star L \sim [0 : 1 : -\frac{4}{3} : \frac{1}{2}]$. Then $(A \star B) \star L \sim [1 : -\frac{14}{5} : -\frac{8}{5} : \frac{8}{5}]$ and $(A \star L) \star (B \star L) \sim [1 : -\frac{14}{5} : -\frac{8}{5} : \frac{8}{5}]$: equal, which is the kei law. But $A \star (B \star L) \sim [1 : \frac{18}{25} : -\frac{24}{25} : -\frac{16}{25}]$, so $\star$ is not associative. Applying L twice returns A.

Why it is fun. The three kei axioms come from the fact that reflection is an isometry of $\langle\, , \rangle$; associativity fails.



Real picture of A, L, $A \star L$.

Example 15. Tangency with complex coefficients

Exercise. Find c so that $y = 2ix + c$ is tangent to K, then compute the double point and $K \star L$.

Solution. Inputs. $K = [1 : 0 : 0 : -1]$, i.e. $x^2 + y^2 = 1$; $L = [0 : 2i : -1 : \sqrt{3}i]$, i.e. $y = 2ix + i\sqrt{3}$. **Numbers.** $Q(K) = 4$, $Q(L) = -3$, $\langle K, L\rangle = -2\sqrt{3}i$, $\kappa = 1$. **Image.** $K \star L = K - \frac{2\langle K,L\rangle}{Q(L)} L \sim [1 : \frac{8\sqrt{3}}{3} : \frac{4\sqrt{3}i}{3} : 3]$, i.e. $x^2 + \frac{8\sqrt{3}x}{3} + y^2 + \frac{4\sqrt{3}iy}{3} + 3 = 0$. **Common points** (finite, of K and L): $\left(-\frac{2\sqrt{3}}{3}, -\frac{\sqrt{3}i}{3}\right)$ — one double point (complex). A line $y = mx + c$ is tangent to K iff $c^2 = 1 + m^2$, which is $\kappa = 1$. For $m = 2i$ this gives $c^2 = -3$, so $c = i\sqrt{3}$. The double point is $(-\frac{2}{\sqrt{3}}, -\frac{i}{\sqrt{3}})$. The image $K \star L$ is tangent to K there too: $\kappa(K \star L, L) = \kappa(K, L) = 1$, by invariance under s_L, which fixes L.

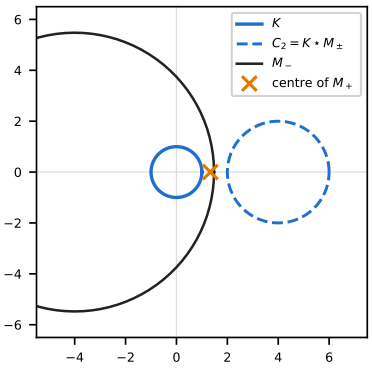
Why it is fun. "Touching" needs no picture: a complex line is tangent to the real unit circle at a point with an imaginary y. (No real picture: every point involved is non-real.)

Example 16. Find the mirror: swap two disjoint circles

Exercise. Find all mirrors M with $K \star M = C_2$ for $C_2 : (x-4)^2 + y^2 = 4$.

Solution. Inputs. $K = [1 : 0 : 0 : -1]$, i.e. $x^2 + y^2 = 1$; $C_2 = [\frac{1}{2} : -4 : 0 : 6]$, i.e. $(x-4)^2 + y^2 = 4$. **Numbers.** $Q(K) = 4$, $Q(C_2) = 4$, $\langle K, C_2\rangle = -11$, $\kappa = \frac{121}{16}$. **Image.** $K \star C_2 = K - \frac{2\langle K,C_2\rangle}{Q(C_2)} C_2 \sim [1 : -\frac{88}{15} : 0 : \frac{128}{15}]$, i.e. $x^2 - \frac{88x}{15} + y^2 + \frac{128}{15} = 0$. **Common points** (finite, of K and C_2): $\left(\frac{13}{8}, -\frac{\sqrt{105}i}{8}\right)$, $\left(\frac{13}{8}, \frac{\sqrt{105}i}{8}\right)$ — conjugate pair. Rescale C_2 to $C_2' = [\frac{1}{2} : -4 : 0 : 6]$ so that $Q(C_2') = Q(K) = 4$ (always possible over $\mathbb{C}$). If $Q(C) = Q(C')$, then $M = C - C'$ gives $\langle C, M\rangle = Q(C) - \langle C, C'\rangle = \frac{1}{2}Q(M)$, so $C \star M = C'$; the same works for $M = C + C'$, since $-C' \sim C'$. Here $M_- = K - C_2' \sim [1 : 8 : 0 : -14]$, the real circle $(x+4)^2 + y^2 = 30$, and $M_+ = K + C_2' \sim [1 : -\frac{8}{3} : 0 : \frac{10}{3}]$, i.e. $(x - \frac{4}{3})^2 + y^2 = -\frac{14}{9}$, a circle of *imaginary radius*. There are no others: $K \star M = K - kM$ forces M into the pencil $K - \lambda C_2'$, and there only $\lambda = \pm 1$ works. They are orthogonal, $\langle M_-, M_+\rangle = 0$, and both pass through the hidden points $(\frac{13}{8}, \pm\frac{\sqrt{105}}{8}i)$. For the tangent pair $(x-3)^2 + y^2 = 4$, M_+ degenerates to the point-circle at the contact point $(1, 0)$. (The "Image" line above is $K \star C_2$, for comparison only.)

Why it is fun. Two disjoint circles have two swapping mirrors, one real and one of imaginary radius; when they touch, the second degenerates to a point.



Real picture: K, C_2 and the real mirror M_-. The imaginary mirror M_+ (centre $\times$) has no real points.