Hidden intersections as a kei of complex circles: mirrors, common points and invariants

Carlo Perassi

October 2026

Theory note accompanying a reply to Inigo Quilez's YouTube Short

Abstract

A real line that misses a circle still meets it in two complex points. We study this phenomenon inside complex Möbius geometry. Generalized circles are points $[a : b : c : d]$ of $\mathbb{P}^3(\mathbb{C})$ with the quadratic form $Q = b^2 + c^2 - 4ad$; reflecting a circle C in a mirror L is the orthogonal reflection $C \star L = C - 2\langle C, L\rangle L/Q(L)$. We prove that $\star$ is a kei (up to the usual projective scale), that its maps generate $PO(4, \mathbb{C})$, and that a circle meets its mirror image exactly where it meets the mirror: $C \cap L$ equals the finite part of $C \cap (C \star L)$ whenever $\langle C, L\rangle \neq 0$. Common points are counted on the quadric $\{Q = 0\} \cong \mathbb{P}^1 \times \mathbb{P}^1$, where the missing ones live over the circular points. The projective invariant $\kappa = \langle C, L\rangle^2/(Q(C)Q(L))$ detects tangency ($\kappa = 1$), equals $\cos^2$ of the angle for real configurations, and can be negative over $\mathbb{C}$. We describe the real structures that decide which points are visible, compositions of two mirrors (including a complex angle), and the inverse problem of finding the mirror that swaps two circles (exactly two, orthogonal). Every proposition is backed by a symbolic or numerical check.

Contents

1 Introduction **2**

2 The space of circles **2**

3 The mirror operation **3**

4 The group generated by mirrors **3**

5 Common points **4**

6 The invariant κ **5**

7 Real structures: what is visible **5**

8 Two mirrors **6**

9 The inverse problem: which mirror swaps two circles? **7**

10 Relation to Quilez's Short **7**

A Phantom loci of other curves **8**

B Verification map **9**

Roadmap. §2–3 set up circle space and the mirror; §4 identifies the group; §5–6 prove the common-points theorem and study κ; §7 explains which points are visible; §8–9 treat two mirrors and the inverse problem; §10 relates all this to Quilez's Short; Appendix A summarizes phantom loci of general curves.

1 Introduction

In the Short *"Intersections are always there"* [1], Inigo Quilez intersects a ray with a circle, keeps the square root when its argument turns negative, and plots real and imaginary parts as the line moves: a line that misses a circle still meets it, at two complex points. This note puts that observation in the setting of complex inversive geometry, where circles, lines, points and "imaginary circles" are all vectors in one space and reflecting in a circle is a linear map.

Conventions. We write the mirror on the *right*: $C \star L$ is "C reflected in L". (The companion note `alt_quandle_note` writes $L \star C$ for the same thing.) All forms are bilinear over $\mathbb{C}$: no complex conjugation appears unless we say so. "Circle" means generalized circle (lines included). Worked examples are in the companion `examples16.pdf`; we cite them as Ex. 1–16.

Verification. Each statement carries a tag such as [P3.1]. The script `theory_verify.py` runs the matching exact SymPy checks or numerical tests and writes `theory_verify_log.txt`. Results quoted from earlier notes are checked by `verify.py`, `verify_inversive.py`, `alt_verify.py` and `examples16.py`. Appendix B lists the logs. Numerical checks are evidence, not proofs; every statement also has a written proof or a precise reference.

2 The space of circles

Definition 2.1. A *(generalized) circle* is a point $C = [a : b : c : d] \in \mathbb{P}^3(\mathbb{C})$, read as the curve $a(x^2 + y^2) + bx + cy + d = 0$ in $\mathbb{C}^2$. It is a *line* if $a = 0$. On $\mathbb{C}^4$ put

$$\langle X, Y \rangle = X^{\mathsf{T}} G Y = bb' + cc' - 2(ad' + a'd), \qquad Q(X) = \langle X, X \rangle = b^2 + c^2 - 4ad,$$

$$G = \begin{pmatrix} 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \end{pmatrix}.$$

For $p = (x_0, y_0) \in \mathbb{C}^2$ let $\mathrm{pt}(p) = [1 : -2x_0 : -2y_0 : x_0^2 + y_0^2]$ (the point-circle at p), and $\infty = [0 : 0 : 0 : 1]$.

Proposition 2.2. *[P2.1] (a) G is non-degenerate; over $\mathbb{R}$ it has signature $(3, 1)$. (b) For $a \neq 0$ the circle has centre $m = (-\frac{b}{2a}, -\frac{c}{2a})$ and $r^2 = |m|^2 - \frac{d}{a}$ (with $|m|^2 = m_1^2 + m_2^2$, no conjugation), and $Q = 4a^2 r^2$. (c) $Q(\mathrm{pt}(p)) = 0$ and $\langle \mathrm{pt}(p), C \rangle = -2\,C(p)$, where $C(p)$ is the left side of the equation at p. So* incidence is orthogonality*: $p \in C \iff \langle \mathrm{pt}(p), C \rangle = 0$. (d) The null quadric $\mathcal{Q} = \{Q = 0\} \subset \mathbb{P}^3$ consists of the points $\mathrm{pt}(p)$, $p \in \mathbb{C}^2$ (those with $a \neq 0$), together with the* ideal points *$a = 0$, $b^2 + c^2 = 0$: the point ∞ and the two lines $\{[0 : 1 : i : \delta]\}$, $\{[0 : 1 : -i : \delta]\}$ through it.*

Proof. (a) $Q = b^2 + c^2 + (a - d)^2 - (a + d)^2$. (b), (c) are direct expansions. (d) If $a \neq 0$, scale to $a = 1$; then $d = (b^2 + c^2)/4$ is the value that makes $[1 : b : c : d] = \mathrm{pt}(-b/2, -c/2)$. If $a = 0$ then $Q = b^2 + c^2$, which vanishes iff $c = \pm ib$; $b = c = 0$ gives ∞. □

Proposition 2.3 (The quadric is $\mathbb{P}^1 \times \mathbb{P}^1$)**.** *[P2.2] With $u = x + iy$, $v = x - iy$, the Segre map $([u_0 : u_1], [v_0 : v_1]) \mapsto [u_0 v_0 : -(u_1 v_0 + u_0 v_1) : i(u_1 v_0 - u_0 v_1) : u_1 v_1]$ is an isomorphism $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathcal{Q}$ sending $(u, v) = (x + iy, x - iy)$ to $\mathrm{pt}(x, y)$. The ideal points are exactly $u = \infty$ or $v = \infty$, and $\infty = (\infty, \infty)$. In these coordinates a circle is $a\,uv + \frac{b - ic}{2} u + \frac{b + ic}{2} v + d = 0$.*

Proof. Nullity and the formulas are direct substitutions (checked symbolically). A smooth quadric surface in $\mathbb{P}^3(\mathbb{C})$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ via its two rulings [5, Ex. 2.21]; the displayed map is that isomorphism for our G. □

Remark 2.4 (Circular points)**.** [P2.3] In the projective plane with coordinates $(X : Y : W)$ every circle $a(X^2 + Y^2) + bXW + cYW + dW^2$

passes through $I = (1 : i : 0)$ and $J = (1 : -i : 0)$, and, if $a \neq 0$, its tangent at I is the isotropic line $X + iY + \frac{b+ic}{2a}W = 0$, i.e. the ideal point $[0 : 1 : i : \frac{b+ic}{2a}] \in \mathcal{Q}$, which is orthogonal to C. So the ideal points of $\mathcal{Q}$ on the I-line record *tangent directions at I* (similarly for J). Heuristically, $\mathcal{Q}$ is $\mathbb{P}^2$ blown up at I, J with the line at infinity contracted to ∞ [5]. Two circles meet in 4 points of $\mathbb{P}^2$ (Bézout); I and J are always among them, and the other two are what we count on $\mathcal{Q}$. For two circles with $a \neq 0$, an ideal common point on the I-line exists exactly when they share their tangent at I (likewise for J) (e.g. concentric circles, Ex. 5). We only use the checked statements above; the blow-up description is quoted, not proved.

3 The mirror operation

Definition 3.1. For $L \in \mathbb{C}^4$ with $Q(L) \neq 0$ (a *mirror*) define $s_L(X) = X - \dfrac{2\langle X, L\rangle}{Q(L)} L$ and $C \star L = s_L(C)$.

Proposition 3.2 (Reflection, closure, involution)**.** *[P3.1] s_L is the orthogonal reflection in the hyperplane $L^\perp$: it fixes $L^\perp$ pointwise, sends $L \mapsto -L$, preserves $\langle , \rangle$, and $s_L^2 = \mathrm{id}$. It is linear in X and unchanged by $L \mapsto \lambda L$, so $\star$ is well defined on $\mathbb{P}^3 \times \{Q \neq 0\}$. Consequently $Q(C \star L) = Q(C)$; the sets $\{Q \neq 0\}$ and $\mathcal{Q}$ are both preserved; and $(C \star L) \star L = C$.*

Proof. $\langle s_L X, s_L Y\rangle = \langle X, Y\rangle - 4\langle X, L\rangle\langle Y, L\rangle/Q(L) + 4\langle X, L\rangle\langle Y, L\rangle Q(L)/Q(L)^2 = \langle X, Y\rangle$. The rest is immediate. □

Proposition 3.3 (Real mirrors)**.** *[P3.2] For a real circle L with centre m and radius $r > 0$, $\mathrm{pt}(p) \star L = \mathrm{pt}\big(m + r^2 \frac{p-m}{|p-m|^2}\big)$: inversion. For a real line it is the Euclidean reflection. For the imaginary circle $x^2 + y^2 = -1$ (real coefficients, $Q = -4$) it is $p \mapsto -p/|p|^2$ (Ex. 9).*

The formula extends these maps to all of $\mathbb{C}^2$, where $|p - m|^2 = (p_1 - m_1)^2 + (p_2 - m_2)^2$ is the complex-bilinear square; at its zeros (points on the isotropic lines through m) the image is an ideal point.

Theorem 3.4 (The circle kei)**.** *[T3.3] On $\mathcal{X} = \{[C] : Q(C) \neq 0\}$, for all A, B, L:*

$$\textit{(K1) } L \star L = L, \qquad \textit{(K2) } (A \star L) \star L = A, \qquad \textit{(K3) } (A \star B) \star L = (A \star L) \star (B \star L).$$

So $(\mathcal{X}, \star)$ is a kei (involutory quandle) [7, 8, 9]; so is its real part. $\star$ is neither commutative nor associative.

Proof. (K1) $s_L(L) = -L$: equality holds *in $\mathbb{P}^3$ only*; as vectors $L \star L = -L$. (K2) is $s_L^2 = \mathrm{id}$. (K3) For any isometry g, $g\, s_B\, g^{-1} = s_{gB}$, since $s_{gB}(X) = X - 2\langle X, gB\rangle gB/Q(B) = g\, s_B(g^{-1}X)$. With $g = s_L$: $s_{B\star L} = s_L s_B s_L$, and applying this to $A \star L = s_L A$ gives (K3). Counterexamples to commutativity and associativity: Ex. 7 versus Ex. 2, and Ex. 14. □

Remark 3.5 (Domain caveats)**.** (i) Point-circles and isotropic lines ($Q = 0$) are acted on but cannot act: clearing denominators gives $Q(L)X - 2\langle X, L\rangle L \sim L$ for every X (Ex. 10, 11). (ii) If $\langle C, L\rangle = 0$ then $C \star L = C$: orthogonal circles fix each other (Ex. 4). (iii) (K1) and (K3) hold up to scale, which is all that $\mathbb{P}^3$ sees.

4 The group generated by mirrors

Theorem 4.1. *[T4.1] The maps s_L ($Q(L) \neq 0$) generate the orthogonal group $O(4, \mathbb{C})$ of $\langle , \rangle$; on $\mathbb{P}^3$ this is $PO(4, \mathbb{C}) = O(4, \mathbb{C})/\{\pm 1\}$. The real mirrors with $Q(L) > 0$ (real circles and lines) generate a group acting on $\mathbb{P}^3(\mathbb{R})$ as $PO(3, 1)$, the Möbius group of the sphere.*

Proof. By the Cartan–Dieudonné theorem, over a field of characteristic $\neq 2$ every isometry of a non-degenerate n-dimensional quadratic space is a product of at most n reflections in non-isotropic vectors [4]. Applied over $\mathbb{C}$ and over $\mathbb{R}$ this gives $O(4,\mathbb{C})$ and $O(3,1)$. For the last claim, reflections in vectors with $Q<0$ are not needed projectively: if $Q(N)<0$, pick an orthogonal basis e_1,e_2,e_3 of the positive definite $N^\perp$; then $s_{e_1}s_{e_2}s_{e_3}s_N=-\mathrm{id}$, so $s_N\equiv s_{e_1}s_{e_2}s_{e_3}$ on $\mathbb{P}^3$. The identification with the Möbius group, generated by inversions, is classical [6]. □

The check [T4.1] decomposes 200 random elements of $O(4,\mathbb{C})$ into 4 reflections, and confirms $s_N=-s_{e_1}s_{e_2}s_{e_3}$ for $N=x^2+y^2=-1$. On $\mathcal{Q}\cong\mathbb{P}^1\times\mathbb{P}^1$ the identity component $PSO(4,\mathbb{C})\cong PSL(2,\mathbb{C})\times PSL(2,\mathbb{C})$ acts factorwise by Möbius maps of u and v, and elements outside it swap the factors (a standard fact we quote, not prove).

5 Common points

Definition 5.1. The *common points* of independent circles C,L are the points of $\mathcal{Q}$ orthogonal to both, i.e. $\mathcal{Q}\cap\mathbb{P}(U)$ with $V=\mathrm{span}(C,L)$, $U=V^\perp$. Their *finite part* is the set of those of the form $\mathrm{pt}(p)$; by Prop. 2.2(c) it is exactly $\{p\in\mathbb{C}^2 : C(p)=L(p)=0\}$, which we write $C\cap L$.

Theorem 5.2 (Common points)**.** *[T5.1] Let C,L be independent with $Q(L)\neq 0$.*

1. *If $\langle C,L\rangle\neq 0$, then $\mathrm{span}(C,C\star L)=\mathrm{span}(C,L)$. Hence C and $C\star L$ have the same common points as C and L, on all of $\mathcal{Q}$; in particular $C\cap L$ is the finite part of $C\cap(C\star L)$.*
2. *If $\langle C,L\rangle=0$, then $C\star L=C$ and (1) says nothing.*
3. *$\mathcal{Q}\cap\mathbb{P}(U)$ is: two distinct points if $\det\mathrm{Gram}(C,L)=Q(C)Q(L)-\langle C,L\rangle^2\neq 0$; one point (a double point, tangency) if the Gram matrix has rank 1; all of the line $\mathbb{P}(U)$ if the Gram matrix vanishes.*

Proof. (1) $C\star L=C-kL$ with $k=2\langle C,L\rangle/Q(L)\neq 0$, so $C\wedge(C\star L)=-k\,C\wedge L$. (2) is Remark 3.5. (3) $U\cap U^\perp=U\cap V=V\cap V^\perp$, so $Q|_U$ and $Q|_V$ have the same rank. $Q|_U$ is a binary quadratic form: of rank 2 it has two distinct isotropic lines, of rank 1 it is a square with one double root, and of rank 0 every vector of U is null. □

Remark 5.3 (Counting and exceptions)**.** [E5.2] "Always two" means two on $\mathcal{Q}$, with multiplicity. The finite count drops by the number of ideal common points:

- *Lines* ($a=0$) all pass through ∞: two lines meet at ∞ and at one finite point (Ex. 6), and parallel lines meet only at ∞, doubly ($\kappa=1$).
- *Concentric circles* share their tangents at I and J: the common points are $[0:1:\pm i:0]$, with no finite point (Ex. 5).
- *Isotropic lines* ($b^2+c^2=0$, $a=0$) have $Q=0$: they are null vectors themselves, they cannot be mirrors, and they meet a circle in at most one finite point (Ex. 10: $y=ix+1$ meets $x^2+y^2=1$ only at $(0,1)$).
- The rank-0 case needs $Q(C)=Q(L)=\langle C,L\rangle=0$, e.g. two point-circles at zero (complex) distance, $\mathrm{pt}(0,0)$ and $\mathrm{pt}(1,i)$, whose whole span is null.
- *Point-circles* are allowed as C but not as L (Ex. 11).

6 The invariant κ

Definition 6.1. For $Q(C)Q(L) \neq 0$ put $\kappa(C,L) = \dfrac{\langle C,L\rangle^2}{Q(C)\,Q(L)}$. It is symmetric and independent of scaling.

Proposition 6.2. *[P6.1]*

1. *$\kappa(gC, gL) = \kappa(C,L)$ for every $g \in O(4,\mathbb{C})$; in particular $\kappa(C \star M, L \star M) = \kappa(C,L)$ and $\kappa(C \star L, L) = \kappa(C,L)$.*
2. *C and L are tangent (one double common point) iff $\kappa = 1$; they are orthogonal iff $\kappa = 0$.*
3. *Real circles with radii r_1, r_2 and centre distance δ: $\langle C_1, C_2\rangle/\sqrt{Q_1 Q_2} = (r_1^2 + r_2^2 - \delta^2)/(2r_1 r_2)$; when they cross at angle θ, $\kappa = \cos^2\theta$. Two real lines at angle θ: $\kappa = \cos^2\theta$. A circle of radius r and a line at distance δ from its centre: $\kappa = \delta^2/r^2$.*
4. *For real C, L with $Q(C), Q(L) > 0$: $\kappa < 1$ gives two real common points, $\kappa = 1$ tangency, $\kappa > 1$ a conjugate ("hidden") pair.*
5. *Over $\mathbb{C}$ every value occurs: $\kappa(x^2 + y^2 = 1,\ y = mx + c) = c^2/(1 + m^2)$. Negative values occur, e.g. $\kappa = -1$ for $(x - i)^2 + y^2 = 1$ and $x = 0$ (Ex. 8), and $\kappa = -\frac{1}{4}$ for a real circle and the imaginary circle $x^2 + y^2 = -1$ (Ex. 9). For real C, L, $\kappa < 0$ forces $Q(C)Q(L) < 0$.*

Proof. (1) g preserves $\langle , \rangle$. (2) With $Q(C)Q(L) \neq 0$, $\det \mathrm{Gram} = Q(C)Q(L)(1 - \kappa)$, so (2) is Theorem 5.2(3); orthogonality is $\langle C, L\rangle = 0$. (3) Direct computation (symbolic); the angle formula is the cosine rule for the triangle formed by the two centres and a crossing point. (4) The Gram determinant $Q_1 Q_2(1-\kappa)$ is positive when V is positive definite and negative when V has signature $(1,1)$. In the first case $U = V^\perp$ has signature $(1,1)$ and two real null lines. In the second U is positive definite, so it has no real null vector; its two null lines are complex conjugate (Prop. 7.1). (5) Direct computation; $\langle C, L\rangle^2 \geq 0$ for real vectors. □

7 Real structures: what is visible

Complex conjugation σ_0 of coordinates acts on circles and preserves $\langle , \rangle$ up to conjugation. Real circles are its fixed points.

Proposition 7.1 (Hidden pairs)**.** *[P7.1] If C and L are real, σ_0 preserves U, so it fixes both common points or swaps them. A swapped pair is a* hidden pair*: two non-real, mutually conjugate points.*

Proposition 7.2 (The phantom hyperbola)**.** *[P7.2] Let $K : x^2 + y^2 = 1$ and translate the line $y = c$ along its normal ("perpendicular translation"). The common points are $(\pm\sqrt{1 - c^2}, c)$. For $|c| > 1$ they are $(\pm is, c)$ with $s = \sqrt{c^2 - 1}$, and the real pairs (s, c) fill the hyperbola $c^2 - s^2 = 1$. That is, $(is, c) \in K \iff c^2 - s^2 = 1$.*

Proposition 7.3 (Limiting points and polarity)**.** *[P7.3] The null points of $V = \mathrm{span}(C, L)$ itself are the* limiting points *of the pencil. For K and $y = c$ they are $(0, c \pm \sqrt{c^2 - 1})$: real when the common points are hidden ($|c| > 1$), and conjugate when they are real. Each limiting point is sent to the other by both $\cdot \star K$ and $\cdot \star L$. The polarity $U \leftrightarrow U^\perp = V$ thus exchanges the common points and the limiting points [2, Prop. 8]. Quilez's visible dots $(\pm s, c)$ are the limiting points rotated by $90°$ about the foot $(0, c)$.*

Proof. $Q(K + \mu L) = 4 + 4c\mu + \mu^2$, whose roots give the point-circles of centre $(0, c \mp \sqrt{c^2 - 1})$. The heights multiply to 1 (inversion in K), and they are symmetric about $y = c$ (reflection in L). The rest is from Theorem 5.2 applied to V instead of U. □

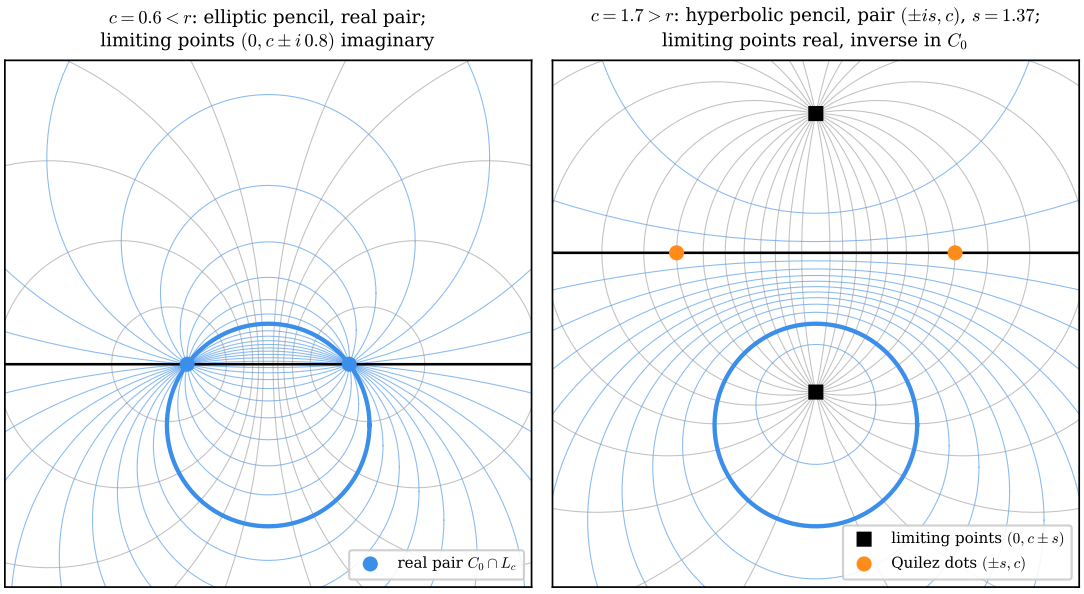


Figure 1: The pencil spanned by K and $y = c$ (thin blue) and its orthogonal pencil (grey). Left, $c = 0.6$: real common points, conjugate limiting points. Right, $c = 1.7$: hidden common points $(\pm is, c)$ and real limiting points $(0, c \pm s)$ (squares); Quilez's dots (orange) are those points rotated by 90° about the foot. From [2].

Proposition 7.4 (Two real forms)**.** *[P7.4]*
$\sigma_1(x, y) = (-\bar{x}, \bar{y})$ is another anti-holomorphic involution preserving circles. Its real circles are $[\alpha : i\beta : \gamma : \delta]$ with $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, on which $Q = -\beta^2 + \gamma^2 - 4\alpha\delta$ has signature $(2, 2)$. So σ_1 defines the real form $PO(2, 2)$ of $PO(4, \mathbb{C})$, next to σ_0's $PO(3, 1)$. The σ_1-real points of K are exactly the phantom hyperbola of Prop. 7.2. The forms are not conjugate, since their real subspaces have different signatures [2].

So the circle and its phantom hyperbola are one complex circle seen through two real structures (Euclidean and Lorentzian). Which common points are visible is a property of the real structure, not of the pair of circles.

8 Two mirrors

Theorem 8.1. *[T8.1] Let M_1, M_2 be independent mirrors with $\kappa = \kappa(M_1, M_2) \neq 1$, $V = \mathrm{span}(M_1, M_2)$ and $T = s_{M_2} s_{M_1}$. Then T is the identity on $U = V^\perp$, preserves V, and $T|_V$ has determinant 1 and trace $4\kappa - 2$. So its eigenvalues on V are $\mu^{\pm 1}$ with $\mu + \mu^{-1} = 4\kappa - 2$; writing $\kappa = \cos^2\theta$ ($\theta \in \mathbb{C}$), $\mu = e^{\pm 2i\theta}$.*

1. *Real lines through a point at angle θ: rotation by 2θ, of finite order iff $\theta/\pi \in \mathbb{Q}$ (Ex. 13: $90°, 60°, 45°$ give orders $2, 3, 4$).*

2. *Parallel lines at distance δ ($\kappa = 1$, excluded above): translation by 2δ.*

3. *Real circles: crossing ($\kappa < 1$) gives an elliptic Möbius map; disjoint ($\kappa > 1$) a hyperbolic one, with $\mu > 0$ real, $\mu \neq 1$.*

4. *Complex angle: for $y = 0$ and $y = mx$ with $m = \tan\theta \in \mathbb{C}$, in $u = x + iy$, $v = x - iy$ the map is $u \mapsto \mu u$, $v \mapsto v/\mu$. For $m = i\sqrt{3}/2$: $\kappa = 4$ and $\mu = 7 \mp 4\sqrt{3} = (2 + \sqrt{3})^{\mp 2}$, a real dilation of u (hyperbolic on the u-line) that sends real points to non-real ones.*

Proof. T fixes $U \subset M_1^\perp \cap M_2^\perp$. With $P_j = M_j M_j^{\mathsf{T}} G/Q(M_j)$, $\operatorname{tr} P_j = 1$ and $\operatorname{tr}(P_2P_1) = \kappa$, so $\operatorname{tr} T = \operatorname{tr}(I - 2P_1 - 2P_2 + 4P_2P_1) = 4\kappa$, i.e. $4\kappa - 2$ on V; $\det T = 1$. Since $\kappa \neq 1$, V is non-degenerate and $\mathbb{C}^4 = U \oplus V$. (1) and (2) are symbolic computations on point-circles. (3) follows from $\mu + \mu^{-1} = 4\kappa - 2$: $|\mu| = 1$ for $0 \le \kappa < 1$ and $\mu > 0$, $\mu \neq 1$ for $\kappa > 1$. (4) The rotation identity of (1) is a polynomial identity in $\cos\theta, \sin\theta$ subject to $\cos^2 + \sin^2 = 1$, so it holds for complex θ; in u, v a rotation by 2θ is $u \mapsto e^{2i\theta}u$, $v \mapsto e^{-2i\theta}v$. The case $m = i\sqrt{3}/2$ was also checked directly. □

Proposition 8.2 (Where two mirrors agree)**.** *[P8.2] For $\kappa \neq 0, 1$ the points $P \in \mathcal{Q}$ with $P \star M_1 = P \star M_2$ are the fixed points of T on $\mathcal{Q}$: the two common points (null lines of U) and the two limiting points (null eigenlines of $T|_V$). Four over $\mathbb{C}$; for real mirrors exactly two of them are real [3, Thm. 5].*

Proof. $P \star M_1 \sim P \star M_2 \iff TP \sim P$. Write $P = u + v$. Then $TP = u + Tv$ is proportional to P iff $v = 0$, or $u = 0$ and v is an eigenvector of $T|_V$. For $\kappa \neq 0, 1$ the eigenvalues are $\neq \pm 1$, so the eigenlines are the two null lines of V. □

9 The inverse problem: which mirror swaps two circles?

Theorem 9.1. *[T9.1] Let C, C' be independent with $Q(C)Q(C') \neq 0$. Rescale C' so that $Q(C') = Q(C) = q$ (possible over $\mathbb{C}$; the two choices differ by sign), and put $p = \langle C, C' \rangle$. A mirror M satisfies $C \star M \sim C'$ iff $M \sim M_\pm := C \pm C'$ and $Q(M_\pm) = 2(q \pm p) \neq 0$. Moreover:*

1. *$\langle M_+, M_- \rangle = 0$: the two mirrors are orthogonal; both lie in the pencil of C, C' and hence pass through $C \cap C'$.*
2. *$Q(M_+)Q(M_-) = 4q^2(1 - \kappa(C, C'))$. If C, C' are tangent ($\kappa = 1$) exactly one candidate is null: it is the point-circle at the contact point, and only one mirror remains.*
3. *If C, C' are real circles, then $M_\pm$ are real. If they cross ($\kappa < 1$), both are real circles or lines (the two bisecting circles); if they are disjoint ($\kappa > 1$), one is a real circle and the other has* imaginary radius *(Ex. 16).*

Proof. If $\langle C, M \rangle = 0$ then $C \star M = C \not\sim C'$. Otherwise $C \star M = C - kM \sim C'$ with $k \neq 0$ forces $M \in \operatorname{span}(C, C')$, say $M \sim C - \lambda C'$. The coefficient of C in $C \star M$ is $1 - 2\langle C, M \rangle / Q(M)$, which must vanish: $Q(M) = 2\langle C, M \rangle$, i.e. $q - 2\lambda p + \lambda^2 q = 2q - 2\lambda p$, i.e. $\lambda^2 = 1$. Conversely, when $\lambda = \pm 1$ and $Q(M) \neq 0$ the image is a multiple of C'. (1) $\langle C + C', C - C' \rangle = Q(C) - Q(C') = 0$. (2) The product is $4(q^2 - p^2)$. If $\kappa = 1$, the null candidate lies in the degenerate pencil and is orthogonal to both C and C' (since $\langle C, C \pm C' \rangle = q \pm p = 0$). So it spans the radical $V \cap U$, which is the double common point. (3) For real C, C' with $q > 0$ the scaling is real; $Q(M_+) + Q(M_-) = 4q > 0$ and the product has the sign of $1 - \kappa$. □

10 Relation to Quilez's Short

Quilez [1] intersects the ray $p = o + \tau d$, $|d| = 1$, with $|p| = r$:

$$p = o - d\big(o \cdot d \pm \sqrt{(o \cdot d)^2 - |o|^2 + r^2}\big),$$

and when the radicand Δ is negative he draws the real part $o - d(o \cdot d)$ (the foot of the perpendicular) and the points "real part ± imaginary part", placing the imaginary part $\pm\sqrt{|\Delta|}$ *along the line d.* The line both rotates and translates, so the trails are tangled. Credit for the construction and the slogan that intersections are always there belongs to him. Video: https://www.youtube.com/shorts/cc-Z_igk4Lo.

Our convention differs in two ways. (i) We fix the direction and translate only, so the hidden points trace a curve: the hyperbola of Prop. 7.2, drawn in $(\operatorname{Im} x, y)$, or in 3D with $\operatorname{Im} t$ as a third

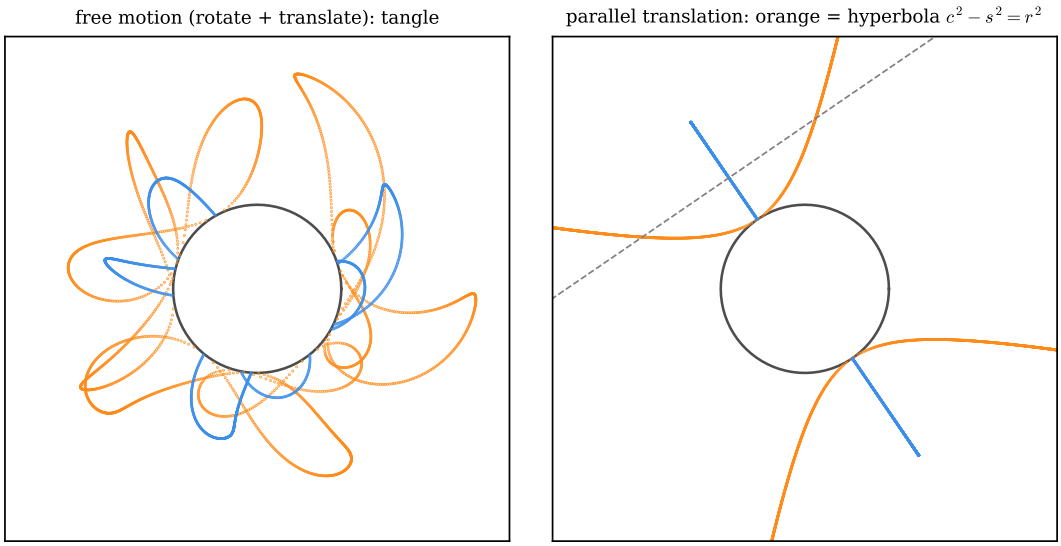


Figure 2: Quilez's construction redrawn ($r = 1$): blue, the real part; orange, real part ± imaginary part drawn along the line. Left: free motion. Right: fixed direction, translation only; the orange trail is the hyperbola $c^2 - s^2 = 1$. From [2].

axis (Appendix A). (ii) We do not fold Im onto the line. Folding is a choice of real slice ($t \mapsto it$), not an operation inside circle space, since $t \mapsto it$ sends circles to hyperbolas (Remark 10.1). His folded dots $hn \pm \sqrt{h^2 - r^2}\, d$ coincide with our hyperbola. They are also the limiting points of the pencil rotated by 90° (Prop. 7.3, Fig. 1). In this sense the visible partner of the hidden pair is a genuine Möbius object. Our claims are about the mathematics in his construction; we do not attribute to him any statement made here.

Remark 10.1 (The quarter turn is not a mirror)**.** The map $R_i : (c, t) \mapsto (c, it)$ in a line frame is not induced by $PO(4, \mathbb{C})$: it sends the circular points to real points at infinity and $x^2 + y^2 = r^2$ to $y^2 - x^2 = r^2$. Inside circle space its role is played by the change of real structure $\sigma_0 \to \sigma_1 = R_i \sigma_0 R_i^{-1}$ (Prop. 7.4). See [2, Thm. 9] for this and the stabilizer of K and a parallel pencil (a Klein four-group).

A Phantom loci of other curves

This appendix summarizes [2]; proofs are given there and checked by `verify.py` ([A.1] re-checks two numbers).

Let $F \in \mathbb{R}[x, y]$ have degree k, $X = \{F = 0\} \subset \mathbb{C}^2$, d a real unit direction, $n = d^\perp$, and $f(c, t) = F(cn + td)$. For real c the line $L_c = \{n \cdot p = c\}$ meets X in the roots of $t \mapsto f(c, t)$, a real polynomial, so they are real or come in conjugate pairs. There are k roots with multiplicity unless d is asymptotic ($F_k(d) = 0$). The *line-real locus* is $X_d = X \cap (\mathbb{R} \times \mathbb{C})$, drawn in $(\operatorname{Re} t, c, \operatorname{Im} t)$; its part with $\operatorname{Im} t \neq 0$ is the *phantom locus* $\Phi_d(X)$.

Theorem A.1 (Tangency theorem [2])**.** *(a) The closure of $\Phi_d(X)$ meets the real plane only at points where $f = f_t = 0$, i.e. tangencies of L_c with $X_\mathbb{R}$ or singular points. (b) At a smooth tangency of contact order m, exactly m smooth branches of X_d meet, at the angles $\pi j/m$ in the t-plane. (c) For $m = 2$, writing $c = c_0 + \frac{\kappa}{2}(t - t_0)^2 + \gamma(t - t_0)^3 + \dots$, the phantom branch is $c = c_0 - \frac{\kappa}{2}(\operatorname{Im} t)^2 + O(|\operatorname{Im} t|^3)$ with $\operatorname{Re} t - t_0 = \frac{\gamma}{\kappa}(\operatorname{Im} t)^2 + O(|\operatorname{Im} t|^3)$: it leaves perpendicular to $X_\mathbb{R}$ with the same curvature, bending the other way.*

(Here κ is a curvature, unrelated to the invariant of §6.) Examples: for the circle the phantom is the hyperbola $c^2 - s^2 = r^2$ in the plane $\operatorname{Re} t = 0$; for $y = x^2$ it is $c = -s^2$; for $y = x^3 - x$ the pairs

have $\operatorname{Re} t \neq 0$ (at $c = 0.6$, $t = -0.6106 \pm 0.3442i$), so the phantom is not planar. The operator $R_\lambda : (c, t) \mapsto (c, \lambda t)$ has stabilizer μ_m on a square-free curve, with $m = \gcd$ of the differences of the exponents of t in f [2, Thm. 5]. The interactive explorer in the repository draws X_d for several curves.

B Verification map

Statement	Check (log)
Props. 2.2, 2.3, Remark 2.4	P2.1–P2.3 (`theory_verify_log.txt`)
Props. 3.2, 3.3; Thm. 3.4	P3.1, P3.2, T3.3
Thm. 4.1	T4.1 (numeric decomposition; theorem quoted)
Thm. 5.2, Remark 5.3	T5.1, E5.2; all 16 examples (`examples16_verify_log.txt`)
Prop. 6.2	P6.1; random real pairs also in `verify_inversive_log.txt`
Props. 7.1–7.4	P7.1–P7.4; `verify_inversive_log.txt`
Thm. 8.1, Prop. 8.2	T8.1, P8.2; `alt_verify_log.txt`
Thm. 9.1	T9.1; Ex. 16
Appendix A	A.1; `verify_log.txt`

References

[1] I. Quilez, *Intersections are always there*, YouTube Short, https://www.youtube.com/shorts/cc-Z_igk4Lo.

[2] C. Perassi, *Where the hidden intersections live: phantom loci of plane curves in* $\mathbb{C}^2$, companion note `writeup.pdf`, 2026.

[3] C. Perassi, *Circles as mirrors: the circle kei, and where two mirrors agree*, companion note `alt_quandle_note.pdf`, 2026.

[4] E. Artin, *Geometric Algebra*, Interscience, 1957 (Cartan–Dieudonné theorem, Ch. III).

[5] J. Harris, *Algebraic Geometry: A First Course*, Springer GTM 133, 1992.

[6] H. Schwerdtfeger, *Geometry of Complex Numbers*, Dover, 1979.

[7] M. Takasaki, Abstractions of symmetric functions, *Tohoku Math. J.* 49 (1943) 143–207.

[8] D. Joyce, A classifying invariant of knots, the knot quandle, *J. Pure Appl. Algebra* 23 (1982) 37–65.

[9] D. Stanovský, The origins of involutory quandles, arXiv:1506.02389.